

Structure and Recognition of 3,4-leaf Powers of Galled Phylogenetic Networks

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Abstract

A graph is a k -leaf power of a tree T if its vertices are leaves of T and two vertices are adjacent in T if and only if their distance in T is at most k . Then T is a k -leaf root of G . This notion was introduced in [13] motivated by the search for underlying phylogenetic trees. We study here an extension of the k -leaf power graph recognition problem. This extension is motivated by a new biological question for the evaluation of the lateral gene transfer on a population of viruses. We allow the host graph to slightly differs from a tree and allow some cycles. In fact we study phylogenetic galled networks in which cycles are pairwise vertex disjoint. We show some structural results and propose polynomial algorithms for the cases $k = 3$ and $k = 4$. As a consequence, squares of galled networks can also be recognized in polynomial time.

1 Introduction

The reconstruction of the evolutionary history of a set of species, based on quantitative biological data, is one of the most challenging problems in computational biology.

Most often the evolutionary history is modeled by an evolutionary tree called phylogeny whose leaves are labeled by species and each internal node represents a speciation event whereby an ancestral specie gives rise to two children. Practically from the biological data one constructs a similarity undirected graph on the set of species where adjacency indicates evolutionary closeness, and then the problem is to build a phylogeny tree from this data. If we understand the distance between species as a distance prescribed (k) in the phylogeny, the reconstruction problem is known as the recognition of k -leaf power graphs [13], for a survey on leaf powers see [3].

Our motivation comes from a question biological question from [2, 1, 12]. They ask us how to measure the relative importance of lateral genes transfer between viruses [7] compare to the normal evolution Darwinian rules. In other words can we infer a tree from the virus biological data or something else (a tree-like structure plus extra edges).

Therefore we consider the problem in which the evolutionary history is modelled by a generalization of trees, the so-called galled networks in which cycles are allowed but under the constraint that they pairwise do not intersect.

In the next sections we first consider some easy facts about phylogenetic networks and in section 4, when studying 4-leaf g-networks we show an important difference with 4-leaf power of trees. Using

maximal clique graphs and block decomposition we propose a polynomial algorithm to recognize these graphs.

2 Preliminaries

It is easy to see that G is a k -leaf \mathcal{N} power iff each of its connected component is. So, we may assume in the following that G is **connected**. The case $k = 2$ is obvious since only sibling leaves can be adjacent, therefore only disjoint union of cliques can be 2-leaf \mathcal{N} power graphs.

Most of the tools developed for the study of k -leaf tree power graphs, are also useful in the generalisation to other classes of phylogenetic networks. Therefore we will extend the notions of basic tree, visible vertex and critical clique as can be found in [4, 5].

A tree or a network is called **basic** if each inner vertex is attached to at most one leaf.

A vertex of N is **visible** if there is a leaf attached to it, otherwise it is **invisible**. For any leaf x of N , let $\mathbf{p}(x)$ (for parent of x) be the inner vertex that x is attached to. For any not leaf u , let $\mathbf{l}(u)$ (for leaf of u) be the leaf attached to u if u is visible, and be empty if u is invisible.

The two following results were already known for k -leaf tree power and can be easily extended to other classes of networks.

Proposition 1 *If G is a k -leaf \mathcal{N} power, then there exists a k -leaf \mathcal{N} root N' of G such that there is a bijection between the critical cliques of G and the set of leaves which are adjacent to the same vertex in N' .*

Proof: Let X be a critical clique of G and $N \in \mathcal{N}$ be one of its roots. Let us suppose that in N , the leaves labelled by X are adjacent to several different visible vertices v_1, \dots, v_m , and denote X_i the set of leaves adjacent to v_i . By moving all X_2, \dots, X_m so that they become adjacent to v_1 , and then delete all new created leaves if there is any, we obtain a network N' . It is obvious that the leaf set of N' and of N is the same, and N' is still a \mathcal{N} network, since we ask the class \mathcal{N} to be hereditary and closed under addition and removing leaves. Let G' be the k -leaf power of N' . We will prove that G' is equal to G .

For any two leaves a, b , if $a, b \notin X$ then their distance in N and in N' are the same. So, if they are connected (resp. non connected) in G , they are also connected (resp. non connected) in G' .

If a is in X but $b \notin X$. Suppose that in G , b is adjacent to a , so it is adjacent with all X because X is a critical clique of G . Then the distance between b and the leaves of X_1 in N are at most k , i.e in N' the distance from b to the leaves of X are at most k . Then, b is adjacent to all X in the G' . The same, if b is not adjacent to X in G , then b is not adjacent to any leaf of X in G' .

If a, b are both in X . Since X is a clique of G , ab is connected in G . They are obviously connected in G' because their distance in N' is exactly 2 (they are attached to the same node v_1). So, G' is equal to G .

By using this construction for any critical clique of G , then for each critical clique X of G , there is a vertex v_X in N' such that all vertices in X are attached to v_X of N' . Suppose that there are two critical cliques X, Y such that $v_X = v_Y$, then it is easy to see that in the k -leaf power of N' , $X \cup Y$ is a critical clique. However, the k -leaf power of N' is G , it is a contradiction because X, Y are critical cliques in G so $X \cup Y$ can not be a critical clique of G . This achieves the proof. \square

Proposition 2 *G is a k -leaf \mathcal{N} power if and only if $CC(G)$ is a k -leaf basic \mathcal{N} power.*

Proof: Suppose that G is a k -leaf \mathcal{N} power and N is one of its roots. By the above proposition, we can choose N such that the leaves of any critical clique of G are attached to the same visible vertex. By contracting each set of leaves of N which adjacent to the same node, we obtain a basic network $N' \in \mathcal{N}$. It is easy to see that N' is a k -leaf power of $CC(G)$.

Conversely, if $CC(G)$ is a k -leaf basic \mathcal{N} network power and N' is one of its basic roots. By replacing each leaf of N' by the set of leaves of the corresponding critical cliques, we obtain a network N which is a k -leaf root of G . \square

3 3-leaf power of \mathcal{N} networks.

It is known that a connected graph G is a 3-leaf tree power if and only if its critical clique graph is a tree [14, 4, 6]. Our next result generalizes this result to \mathcal{N} network powers.

Theorem 1 *A graph G is a 3-leaf \mathcal{N} power if and only if its critical clique graph belongs to \mathcal{N} .*

Proof: Using previous Proposition 2, let us suppose that $CC(G) = (V, E)$ is a 3-leaf \mathcal{N} power graph and let N be one of its roots. By definition it exists a bijective mapping θ from V to the leaves of N .

For every t, t' leaves of N $dist(t, t') \leq 3$ iff $p(t)p(t')$ is an edge of N . Therefore $\theta^{-1}(t)\theta^{-1}(t') \in E$ iff $p(t)p(t')$ is an edge of N . So $CC(G)$ is isomorphic to a subgraph of N . We have forced the class \mathcal{N} to be hereditary, so $CC(G)$ belongs to \mathcal{N} .

Conversely, if $CC(G)$ belongs to \mathcal{N} , let us build a phylogenetic network N from $CC(G)$ by adding to every $x \in V$ a leaf $l(x)$ attached to x . \mathcal{N} is supposed to be closed by adding leaves, so $N \in \mathcal{N}$. Furthermore we immediately have $d_N(l(x), l(y)) = 3$ iff $xy \in E$. \square

Corollary 1 *The recognition of 3-leaf \mathcal{N} power is equivalent (same complexity) of the recognition problem for \mathcal{N} .*

Corollary 2 *3-leaf g-network powers can be recognized in linear time.*

Proof: Since it is well-known that computing twins (false or true) can be done in linear time see [10], therefore calculating $CC(G)$ can be done in linear time. Recognition of g-networks can also be done in linear time simply by using a depth-first search traversal and computing the block (2-connected components) decomposition of the graph. To conclude we just have to check if every block contains at most one cycle and if these cycles intersect. \square

4 4-leaf basic g-network powers

For $k = 4$ the problem becomes harder to solve and therefore for technical purpose we choose to consider first a slight generalization of trees namely g-networks.

Definition 1 *Let \mathcal{L} be a set of n species. A g-network on \mathcal{L} is a connected undirected graph containing n vertices of degree 1 which are bijectively labelled by \mathcal{L} and in which induced cycles are pairwise vertex-disjoint.*

Moreover, by computing $CC(G)$, one can easily transform a 4-leaf g-network power to a 4-leaf basic g-network power. So, we study the problem only on basic models.

4.1 Preliminaries

As a first remark we notice that if G is a 4-leaf basic g-network power then it has a root N which does not contain any two adjacent invisible vertices. This is simply because if there were an edge between two invisibles vertices, this edge cannot be used to connect 2 leaves via a path of length ≤ 4 . Therefore this edge can be deleted in N . So we can suppose that our networks do not contain two adjacent invisible vertices. This assumption implies that for any two vertices x, y of G , there is a path from x to y in G iff there is a path from $\theta(x)$ to $\theta(y)$ in N .

Definition 2 For a subset of vertices M of G let us define for a root N of G , $N[M]$ to be the minimal connected induced subgraph of N whose leaves are exactly $\theta(M)$ and for which $\forall x, y \in M, d_{N[M]}(\theta(x), \theta(y)) = d_N(\theta(x), \theta(y))$.

When N is known, for every subset of vertices M , $N[M]$ is uniquely defined and easy to compute.

The family \mathcal{F} of g-networks described in Figure 1 is made up with an infinite family of visible, invisible (quasi) stars together with 5 networks N_4 , N_5 , N'_5 , N''_5 , N_6 based on cycles of length 4 to 6.

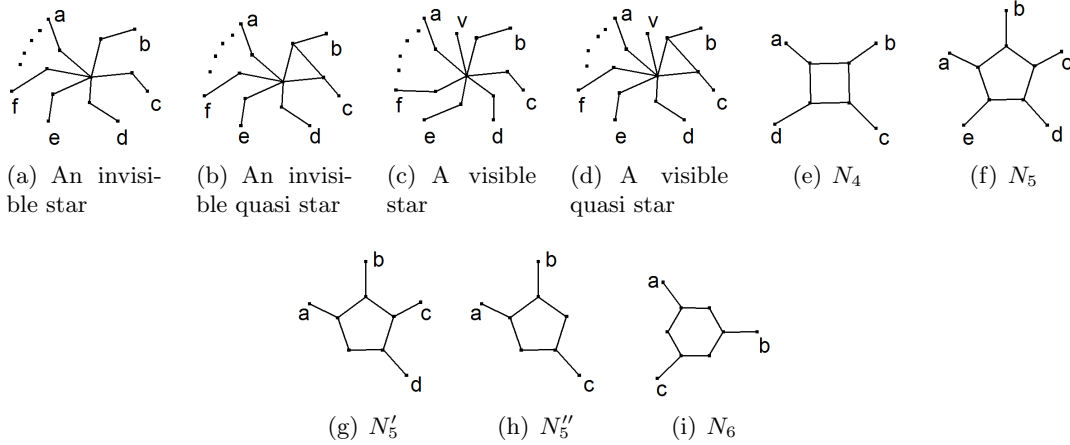


Figure 1: \mathcal{F} the family of roots of maximal clique for 4-leaf basic g-networks power

Observation 1 Suppose that G has a 4-leaf basic g-network root N . Then M is a maximal clique of G iff $N[M]$ is isomorphic to one of the subgraphs in \mathcal{F} .

We denote i -cycle for a chordless cycle of length i . So, N_4 is a 4-cycle without invisible vertex, N_5 is a 5-cycle without invisible vertex, N'_5 is a 5-cycle with one invisible vertex, N''_5 is a 5-cycle with two invisible vertices and N_6 is a 6-cycle with 3 visible vertices of pairwise distance 2.

An *invisible star* (Figure 1(a)) consists of an invisible vertex u , called the *middle vertex* of it, and all visible vertices adjacent to u as well as the leaves attached to these visible vertices. Two visible vertices of an invisible star can be adjacent, in this case we have an *invisible quasi star* (Figure 1(b)). Because there is no intersecting cycle in the constructing networks, there are at most two adjacent visible vertices in each invisible quasi star.

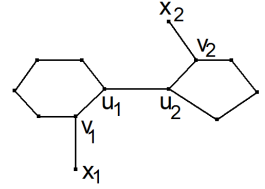
A *visible star* (Figure 1(c)) consists of a visible vertex u , called the *middle vertex* of it, the leaf of u and all visible vertices adjacent to u as well as the leaves attached to these visible vertices. If there are two adjacent visible vertex u, u' such that they are not adjacent to any other visible vertex, then the visible star of u is exactly the visible star of u' , we can consider either u or u' as the middle vertex of this visible star. Similarly we define also *visible quasi star* (Figure 1(d)).

Moreover, if a visible or invisible star is totally included in another visible or invisible star, or a network $N_4, N_5, N'_5, N''_5, N_6$ then we do not consider it as a visible or invisible star. For example, this is the case of the visible star whose middle vertex is $p(b)$ in Figure 1(e) when $p(b)$ is not adjacent to any other visible vertex different from $p(a), p(c)$; or the invisible stars having $\{a, d\}$ as the leaf set in Figure 1(g) when the middle vertex of this star is not adjacent to any other visible vertex different from $p(a), p(d)$.

Proof: \Rightarrow Let $M = \{x_1, \dots, x_k\}$ be a maximal clique of G , then in N we have $d(x_i, x_j) \leq 4$ for any i, j .

We first prove that $N[M]$ cannot contain more than one cycle. In fact, if there are two cycles C_1, C_2 in $N[M]$, they are vertex disjoint and be separated by a path containing at least one edge.

Let u_1, u_2 be the extremities of this path on C_1, C_2 . There must exist a visible vertex v_i on C_i different from u_i for any $i = 1, 2$. Indeed, suppose otherwise, then in order that the distance between every pair of leaves in M are at least 4, all leaves of M are attached to either u_1 or u_2 . It implies by the definition of $N[M]$ that $N[M]$ consists of only the path from u_1 to u_2 but not the cycles C_1, C_2 , a contradiction. Let x_i be the leaf attached to v_i . Hence, $d_N(x_1, x_2) \geq 5$, it means that they are not connected in G , a contradiction.



If $N[M]$ contains a cycle, that cycle has at most 6 vertices. Otherwise there are two leaves whose distance is greater than 4. So, the Figures 1(b), 1(d), 1(e), 1(f), 1(g), 1(h), 1(i) correspond to all the possible configurations of $N[M]$ in this case.

If $N[M]$ does not contain any cycle, then in order that $d(x_i, x_j) \leq 4$ for any i, j , $N[M]$ must be a star (visible or invisible).

\Leftarrow It is obvious that the 4-leaf power of any network of \mathcal{F} is a clique. By definition, not any network of \mathcal{F} is included in another one. So each one correspond to a maximal clique of G . \square

Therefore, one can construct a 4-leaf g-network leaf of a graph G by replacing each maximal clique M of G by a network in the family \mathcal{F} in an appropriate way.

It should be noticed that the networks N_4, N_5, N'_5, N''_5 and N_6 of the family \mathcal{F} are not chordal graphs. Nevertheless maximal cliques still play a great role as will be shown in the following sections.

Definition 3 (Maximal cliques graph) Let G be a biconnected graph. The maximal cliques graph of G , denoted by $MC(G)$ is a graph whose nodes correspond to maximal cliques of G . Two nodes A, B are connected by an edge if and only if the corresponding maximal cliques of A and B have common vertices, and furthermore $A \cap B$ is maximal with respect to A or B . ($A \cap B$ is maximal with respect to A if there is not any node C such that $A \cap C$ is included in $A \cap B$).

The label of a node A of $MC(G)$, denoted by $l(A)$, is equal to the vertex set of the maximal clique A . The label of an edge AB , denoted by $l(AB)$, is equal to the common vertex set of the

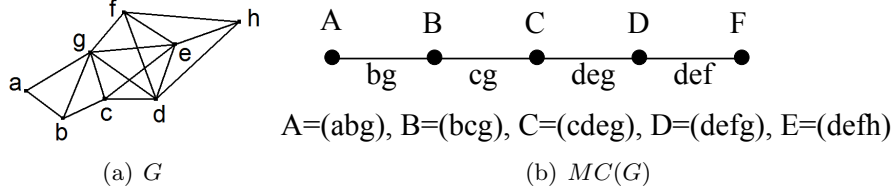


Figure 2: An example of maximal cliques graph

two maximal cliques A, B . The *label* of a subgraph S , denoted by $l(S)$, is equal to the union of the labels of the edges of S . The *weight* of an edge of $MC(G)$ is the cardinal of its label. The *size* of a node is the cardinal of its label.

Let A be a node of $MC(G)$, denote by $N[A]$ the induced subgraph of N as defined in Definition 2, on the maximal clique of G corresponding to the node A . For the sake of simplicity, when we represent $N[A]$ in figures, except the leaves of A which have concrete labels, we denote the labels of the other leaves by A .

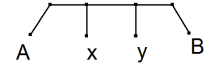
We say that a network n is a 4-leaf root of a subgraph S of $MC(G)$ iff $MC(n^{4l}) = S$.

Let S be a subgraph of $MC(G)$, then denote by $G[S]$ the induced subgraph of G on the vertex set consisting of all $l(A)$ for every node A of S . If a network n is a root of $G[S]$, we also say that it is a root of S .

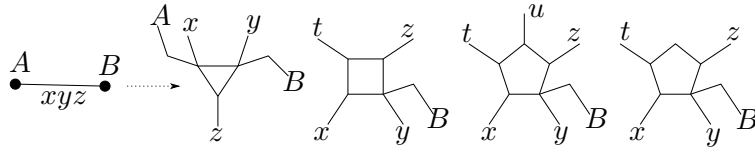
With this definition of $MC(G)$, we can deduce several properties of $MC(G)$ related to the family \mathcal{F} in the observation below.

Observation 2 Let N be a 4-leaf basic g -network root of G .

(i) If AB is an edge of $MC(G)$ such that $l(AB) = \{x, y\}$ and $N[A], N[B]$ are visible (quasi) star, then x, y are the two leaves attached to the middle vertices of $N[A]$ and $N[B]$.

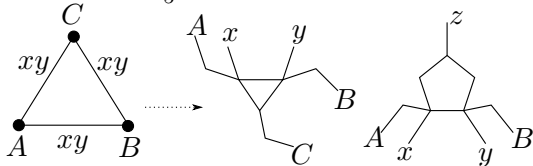


(ii) If AB is an edge of $MC(G)$ such that $|AB| = 3$ then either $N[A], N[B]$ are visible quasi stars sharing a same triangle or one of them is N_4 or N'_5 or N''_5 and the other is a visible star.

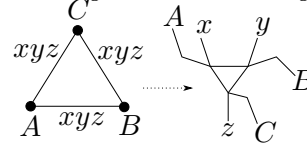


(iii) If ABC is a triangle of $MC(G)$, then:

- either $|AB| = |BC| = |AC| = 2$, then one of $N[A], N[B], N[C]$ is either an invisible quasi stars or a N''_5 .



- or $|AB| = |BC| = |AC| = 3$, then $N[A], N[B], N[C]$ are 3 visible quasi stars sharing the same triangle.

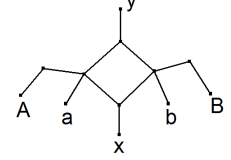


(iv) If AB is an edge of $MC(G)$ such that $|AB| = 1$ then either $N[A]$ or $N[B]$ is an invisible (quasi) star.

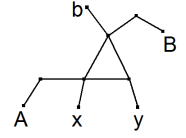
(v) If MA, MB are two edges of $MC(G)$ such that $|MA| = |MB| = 3$ and AB is not an edge of $MC(G)$ then $N[M]$ is either N_4 or N_5 or N'_5 and $N[A], N[B]$ are visible stars.

Proof: (i) We check all the possible configurations of $N[A], N[B]$ such that they have two common leaves x, y .

Suppose these common vertices are not middle vertices of $N[A], N[B]$, and the middle vertices of $N[A], N[B]$ are respectively a, b . So, there is a 4-cycle N_4 with the leaves x, y, a, b . Let C be the maximal clique corresponding to this cycle. Then $A \cap C = \{x, a, b\}$ and $B \cap C = \{y, a, b\}$, i.e. $A \cap B$ is include in $A \cap C$ and $B \cap C$. By definition of $MC(G)$, AB is not an edge of $MC(G)$, a contradiction.



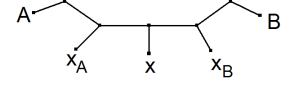
Suppose that x is the label of the middle vertex of $N[A]$, but y is not the label of the middle vertex of $N[B]$. Because x is contained in the star $N[B]$, so b is also in the star having $p(x)$ as middle vertex. In other words, b is contained in $N[A]$, i.e. A, B have 3 common vertices x, y, b , a contradiction. So, x, y are the labels of the middle vertices of $N[A], N[B]$.



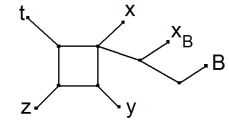
(ii,iii) It is easy to check theses claim by considering all possible configurations of $N[A], N[B]$, (and $N[C]$) with the condition of vertex-disjoint cycles in N .

(iv) Suppose that neither $N[A]$ nor $N[B]$ is an invisible (quasi) star, and let x be the unique common leaf of $N[A]$ and $N[B]$. By considering all other possible configurations of $N[A]$ and $N[B]$, we can deduce that it is not possible.

For example suppose that $N[A], N[B]$ are visible (quasi) stars. Let x_A, x_B be the leaves of the middle vertex of $N[A]$ and $N[B]$. Let consider the visible (quasi) star having $p(x)$ as its middle vertex, it corresponds to the maximal clique C . So, C contains x_A, x_B . We deduce that $A \cap B \subset A \cap C$ and $A \cap B \subset B \cap C$. By definition of $MC(G)$, AB is not an edge, a contradiction.



If $N[A]$ is visible (quasi) star, $N[B]$ is N_4 , let x_A be the leaf of the middle vertex of $N[A]$. Let consider the visible (quasi) star having $p(x)$ as its middle vertex, it corresponds to the maximal clique C . So, C contains x_B and y, t . We deduce that $A \cap B \subset A \cap C$ and $A \cap B \subset B \cap C$. By definition of $MC(G)$, AB is not an edge.



Similarly, we can check that in any other case, there is always a maximal clique C such that $A \cap B \subset A \cap C$ and $A \cap B \subset B \cap C$, a contradiction.

(v) We have $|MA| = 3$, so by Observation (ii):

- either $N[A], N[M]$ are visible quasi stars sharing a triangle. Since $|MB| = 3$, $N[B]$ must be also a visible quasi star which share the same triangle with $N[M]$ and $N[A]$. So, AB is connected by an edge of weight 3 in $MC(G)$, a contradiction.

- or $N[M]$ is N_4, N_5 , or N'_5 and $N[A], N[B]$ are visible stars, so we are done. \square

Lemma 1 If G is a 4-leaf basic g -network power, then

(i) each edge of G is contained in at most 3 maximal cliques.

(ii) any two maximal cliques have at most 3 vertices in common.

Proof: Let N be a 4-leaf basic g-network root of G .

(i) Let xy be an edge of G which is contained in a maximal clique A .

- If $N[A]$ is not a visible or invisible quasi star: suppose that there are other two maximal cliques B, C that contain xy . Then for any configuration of the family \mathcal{F} that $N[A], N[B], N[C]$ take, we can check that $N[A \cup B \cup C]$ contains always two non disjoint cycles, contradicting that N is a g-network. So, in this case, there are at most 2 maximal cliques containing xy .

- If $N[A]$ is a visible or invisible quasi star: then there are at most two other maximal cliques different from A that can contain x, y by the same reason. In the case that there are exactly 2 other maximal cliques B, C that contain x, y , then ABC is a triangle of $MC(G)$.

(ii) Let A, B be two maximal cliques having more than 1 common vertices. Always by the condition of vertex-disjoint cycles in N , we deduce that if both $N[A], N[B]$ are either visible stars or invisible (quasi) stars or N_5'' , or N_6 , then they have at most 2 common leaves. If one of them is a visible quasi star or N_4 or N_5 , or N_5' , then they have at most 3 common leaves. Moreover, when they have exactly 3 common vertices, either one of them is N_4 or N_5 or N_5' , or both of them are visible quasi stars (Claim (ii) of Observation 2). \square

Corollary 3 For $G = (V, E)$ a 4-leaf basic g-network power, the number of maximal cliques of G is bounded by $3|E|$.

Theorem 2 For $G = (V, E)$ a 4-leaf basic g-network power graph, $MC(G)$ can be computed in $O(|V| \cdot |E|^2)$.

Proof: It suffices to use any algorithm that generates all maximal cliques of G in $O(|V| \cdot |E|)$ delay using polynomial space as described in [8]. If it has more than $3 \cdot |E|$ maximal cliques, we can stop the algorithm and conclude that it is not a 4-leaf basic g-network power, otherwise we can calculate the edges of $MC(G)$ within the same complexity. \square

When a clique is a block, let us call it a *block clique*. Another similar property with 4-leaf basic tree roots is stated in the following lemma.

Lemma 2 If G is a 4-leaf basic g-network power, then there exists a 4-leaf basic g-network root N of G such that for any block clique \mathcal{B} of G , $N[\mathcal{B}]$ is an invisible star.

Proof: Let $\mathcal{B}_1, \dots, \mathcal{B}_m$ be the blocks of G . Let N_0 be a 4-leaf basic g-network root of G . For any \mathcal{B}_i , let $N_i = N_0[\mathcal{B}_i]$. By the above analysis, N_0 is obtained by combining N_i for all $i = 1, \dots, m$. For any block clique \mathcal{B}_i we replace N_i by the invisible star S_i having the vertices of \mathcal{B}_i as its leaf set such that if N_i, N_j have x and $p(x)$ as common vertices, then S_i and N_j also have x and $p(x)$ as common vertices. Let the resulting network is N . It is obvious that S_i is a 4-leaf basic g-network root of \mathcal{B}_i . We will prove that N is a 4-leaf root of G . It is easy to check that for any two leaves x, y of G , if x, y are not in the same block, then their distance in N are greater than 4. This corresponds to the fact that x, y are not connected in G . If x, y are contained in one block and this block is not a clique, then their distance in N is exactly their distance in N_0 . If this block is a clique then their distance in N is exactly 4 because they are two leaves of an invisible star. That corresponds to the fact that xy is connected in G because they are in the same clique. Therefore, G is a 4-leaf power of N .

It is obvious that N is a basic network. We must prove furthermore that the cycles in N are pairwise vertex-disjoint. From the assumption that there is no adjacent invisible vertex in N , we

deduce that each cycle of N is contained in the root of a block of G . Since N_0 is a g-network, the cycles in each N_i are pairwise vertex-disjoint. The invisible stars S_i do not contain any cycle, so the cycles in N are always pairwise disjoint, i.e. N is a basic g-network. \square

4.2 Adding extra constraints to the root graphs

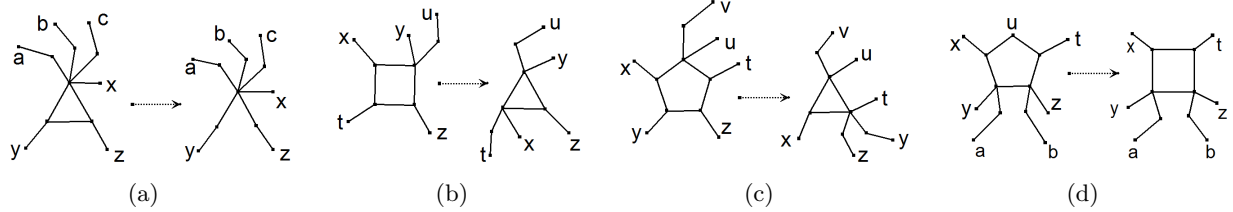


Figure 3: Examples of the simplification

Let us suppose that there is a root N in which there exists a 3 or 4 or 5-cycle having only one vertex which is adjacent to other inner vertex not on the cycle. An easy observation shows that in such a case we can simplify the root to another simpler network (in each case the cycle is shortened) without changing the 4-leaf power and the properties of disjoint cycles of the network as illustrated in Figures 3(a), 3(b), 3(c). If there is a 5-cycle with one invisible vertex u as in Figure 3(d) such that $p(x), p(t)$ and u are not adjacent to any other inner vertex not on the cycle, then we can replace it by the 4-cycle in the right. We apply such a transformation at most once for each cycle.

Therefore we can suppose that the networks that we construct are connected, do not have any two adjacent invisible vertices and do not contain any such 3, 4, 5-cycle. Moreover, according to Lemma 2, then induced subgraph of the roots on each block clique of G is an invisible star.

4.3 Sketch of the analysis of the 4-leaf case

We consider separately each block of G . The steps of the algorithm on one block is resumed in the example of Figure 4. Firstly, we calculate the maximal cliques graph $MC(\mathcal{B}_i)$ of each block \mathcal{B}_i . Next, we find all particular subgraphs of $MC(\mathcal{B})$ such that every root of each one has exactly one cycle. Some conditions must be taken into account on the root set of these subgraphs in order that they are compatible with the others in the same block as well as in the adjacent blocks. After this step, we have the set of cycles of a 4-leaf basic g-network of G , namely cycle-root set \mathcal{R}_c . Therefore, by separating the corresponding subgraphs from \mathcal{B}_i the obtain graph $\mathcal{B} \setminus \mathcal{R}_c$ must be the square of a forest (Theorem 5). The later problem is known to be linear. So, it is easy to construct a root of $\mathcal{B} \setminus \mathcal{R}_c$ which is compatible with the cycle-root set \mathcal{R}_c . So, by combining them together, we obtain a 4-leaf basic g-network of \mathcal{B}_i . The final step is combining the root of each block to have the corresponding root of G .

Hence, recognition of 4-leaf basic g-network powers can be done in polynomial time. As a consequence, squares of basic g-networks can also be recognized in polynomial time.

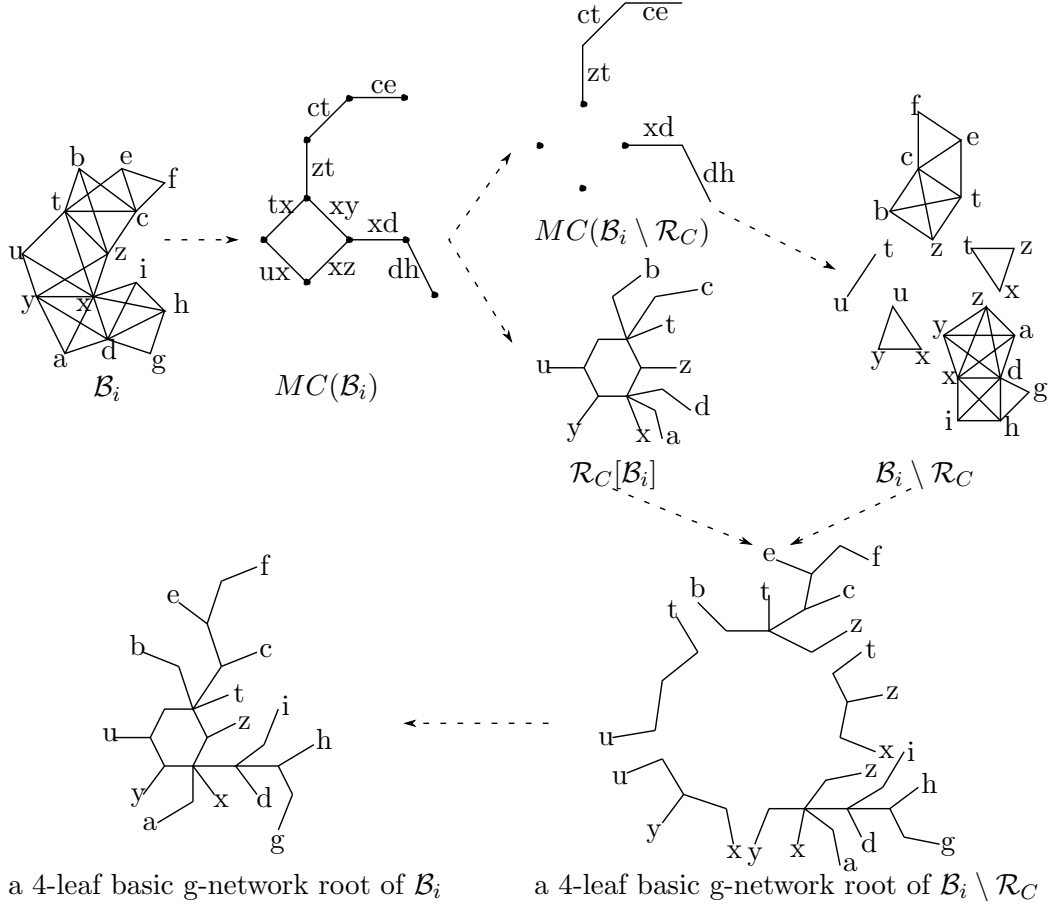


Figure 4: Construction a 4-leaf basic g-network root of a block \mathcal{B}_i of G

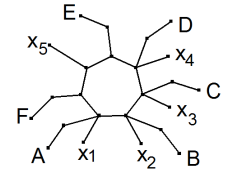
Almost the remaining of this paper is devoted to construct the cycles in the roots of G (Section 4.4). For this purpose, we introduce some special subgraphs of $MC(G)$ namely $\mathcal{C}_e(G)$, $\mathcal{C}_o(G)$ (when G is biconnected) and $\mathcal{C}_c(G)$ (when G is not biconnected).

4.4 Detecting the cycles in the roots of G .

We call a cycle is *big* if it contains at least 7 vertices, otherwise it is *small*. It should be noticed that all cycles that we mention here (cycles of $MC(G)$ or cycles of networks) are *chordless cycles*. We will denote the cycle in N by the lower-case letter c and the cycle in $MC(G)$ by the upper-case letter C .

Let N be a basic g-network. For any cycle c of N , let us define the set of labels of the visible vertices of c as $l(c) = \{x \in \mathcal{L} \mid p(x) \in c\}$. Denote by $S(c)$ the subnetwork of N consisting of c and all (quasi) visible and invisible stars having a vertex of c as their middle vertices.

Example 1 In the figure in the right, we have a cycle c and $S(c)$. By definition, $l(c) = \{x_1, x_2, x_3, x_4, x_5\}$.



Lemma 3 *If G is biconnected, is not a clique and has a 4-leaf basic g -network root N , then:*

(i) *Any invisible vertex of N is contained in a cycle.*

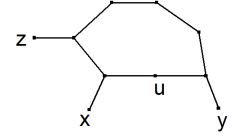
For any cycle c of N and for any vertex u on c :

(ii) *If u is visible then u is not adjacent to any invisible vertex not on c . Furthermore, if u is adjacent to only invisible vertex on c , then u is not adjacent to any other inner vertex not on c .*

(iii) *If u is invisible, let s be the (quasi) invisible star having u as its middle vertex, then for any visible vertex v of s which is not on c , v is not adjacent to any other inner vertex different from u .*

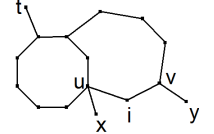
Proof: (i) Suppose that N has an invisible vertex u which is not contained in any cycle.

Let x, y be two leaves of the invisible star of u , and let z be a leaf not in this invisible star which has distance at most 4 with x . Since G is biconnected, there is at least a path in G from y to z which does not pass x . We deduce that in N there is at least a path from $p(y)$ to $p(z)$ that does not pass $p(x)$. In other words, there is a cycle passing u , a contradiction.

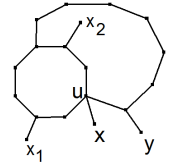


(ii) Let c be a cycle of N and let u be a visible vertex on c .

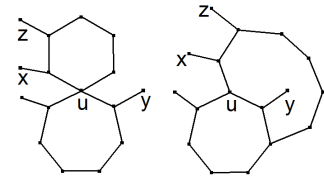
Suppose that u is adjacent to an invisible vertex i not on c . Let v be another vertex adjacent to i , so v is visible. Let x, y be the leaves of u, v . Let t be another leaf such that $p(t)$ is on c . If in N all path from t to y passes by u , then all paths from t to y in G must pass x . This is a contradiction because G is biconnected. So there is another path from $p(t)$ to $p(y)$ which does not pass u , this means that there is another cycle different from c intersecting with c . This is impossible because N is a g -network.



Suppose that u is adjacent to 2 invisible vertices on c , let x_1, x_2 be the leaves of these two invisible vertices. Suppose that u is adjacent to a vertex v not on c . As showed above, v must be visible, let y be the leaf of v . In order that y is contained in the same block with x, x_1, x_2 , there must exist a path from y to x_1 in G which does not pass x . It implied that in N there exists a path from $p(y)$ to $p(x_1)$ which does not pass $p(x)$. However, that will create a cycle intersecting with c , a contradiction.



(iii) Let c be a cycle of N and let u be an invisible vertex on c . Let x be a leaf of the star s of u such that $p(x)$ is not on c . Let y be a leaf of this star such that $p(y)$ is on c . Suppose that $p(x)$ is adjacent to an inner vertex different from u , then there is a leaf z not in s such that $d_{N_0}(x, z) \leq 4$. Then, in G there is a path from z to y which passes x . Because G is biconnected, there must be a path from z to y in G which does not pass x . That will produce in N another cycle intersecting with c at u (there are two cases indicated in the figure), a contradiction. \square



We introduce a list of subgraphs C_1, \dots, C_9 of $MC(G)$ that will be used in the sections below (Figure 5) to detect the cycles in the roots of G .

C_2 is a star of $MC(G)$ consisting of at least two edges such that all of its edges have weight 3 and non of them belongs to a subgraph C_1 .

C_6 is a cycle which is not contained in any subgraph C_3 .

C_9 is a segment of weight 3 which is not contained in any subgraph C_1 or C_2 .

C' is a cycle of $MC(G)$ which is different from C_1, C_6 and C_8 .

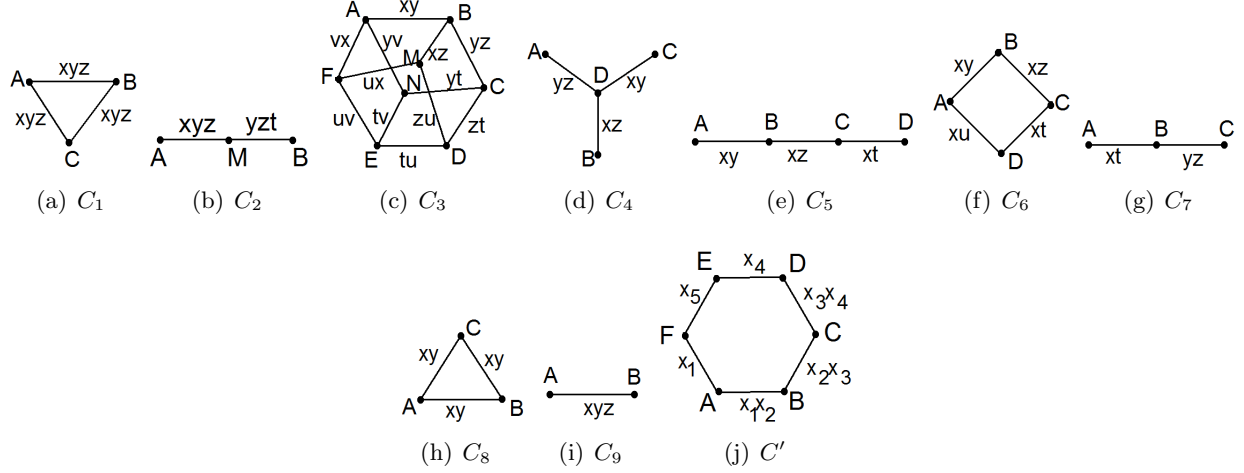


Figure 5: Some subgraphs of $MC(G)$ that imply cycles in the roots of G

The other subgraphs have exactly the forms described in the corresponding figures.

The following lemma states a relation between the cycles of $MC(G)$ and the cycles in the 4-leaf basic g-network roots of G .

Lemma 4 *Suppose that G has a 4-leaf basic g-network root N . Then for any cycle C of $MC(G)$, which is not a 4-cycle in any subgraph C_3 of $MC(G)$, there is a cycle c in N such that $l(C) = l(c)$.*

And for any subgraph S isomorphic to C_3 of $MC(G)$, there is a 6-cycle without invisible vertex c in N such that $l(S) = l(c)$.

Proof: Let $C = [A_1, \dots, A_k, A_1]$ be a cycle in $MC(G)$. Let us denote the label of $A_i A_{i+1}$ by l_i .

Using Observation 1, N can be constructed by replacing each A_i by a subnetwork of \mathcal{F} along the cycle in a way such that if A_i, A_{i+1} has l_i in common then their corresponding subnetworks also have the leaves l_i in common. Since all the graphs in \mathcal{F} are connected, that creates a cycle c in N .

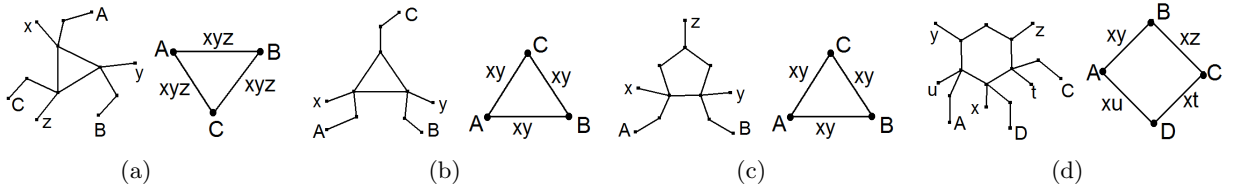


Figure 6: Some small cycles of N and the corresponding cycles in $MC(G)$

So if there is an A_i such that $N[A_i]$ contains a cycle, then this cycle intersects with c . Since the cycles in N must be pairwise disjoint, in this case the cycle of $N[A_i]$ must be exactly c . As indicated in Observation 1, the subnetworks of \mathcal{F} contain cycles of at most 6 vertices. By taking the 4-leaf power of all possible configurations of $S(c)$, there are only 4 cases that the corresponding

maximal cliques graph is a cycle: A 3-cycle without any invisible vertex (Figure 6(a)), a 3-cycle with exactly one invisible vertex (Figure 6(b)), a 5-cycle with exactly two invisible vertices (Figure 6(c)), a 6-cycle with one invisible vertex (Figure 6(d)). It can be verified that in all these cases that $l(c)$ equals $l(C)$.

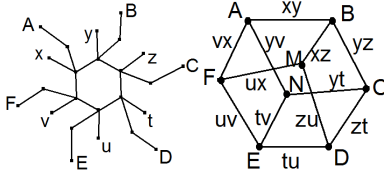


Figure 7: A 6-cycle without invisible vertex and its maximal cliques graph.

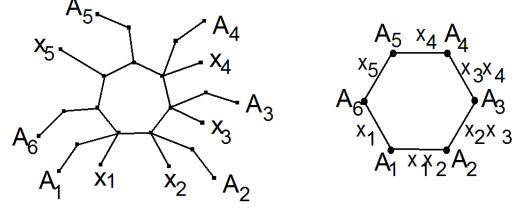


Figure 8: $N[A_i]$ is either a visible or invisible star.

Especially, for the case of 6-cycle without any invisible vertex, the corresponding maximal cliques graph is isomorphic to the graph C_3 (Figure 7). For the other cases, there is no cycle in the corresponding maximal cliques subgraphs.

On the other hand, if there is not any A_i such that $N[A_i]$ contains a cycle, then $N[A_i]$ is a visible star or an invisible star for all i . By replacing each node A_i by a visible or invisible star having A as the leaf set such that if the two maximal cliques have x, y (or x) in common then the corresponding stars also have two leaves x, y (or x) in common, we obtain a cycle c having $l(c)$ such that $l(c)$ is equal to $l(C)$ (Figure 8). \square

This lemma shows that $MC(G)$ can contain chordless cycles or G can contain chordless cycles. This makes a big difference with the k -leaf tree power graphs which were proved to be strongly chordal [4].

4.4.1 Easy cycles when G is biconnected

They consist of all the big cycles, some 6-cycles with 2 invisible vertices, and some small cycles without invisible vertices.

Let us denote by $\mathcal{C}_1(G)$ the set of subgraphs isomorphic to C_1 of $MC(G)$. So it contains all triangles of $MC(G)$ with 3 edges of the same label weighted by 3.

$\mathcal{C}_2(G)$ the set of all subgraphs C_2 of $MC(G)$.

$\mathcal{C}_3(G)$ the set of all subgraphs isomorphic to C_3 of $MC(G)$.

$\mathcal{C}'(G)$ the set of cycles of $MC(G)$ which are different from C_1 , C_6 and C_8 .

Let $\mathcal{C}_e(G) = \mathcal{C}'(G) \cup \mathcal{C}_1(G) \cup \mathcal{C}_2(G) \cup \mathcal{C}_3(G)$ (e for easy).

Lemma 5 Suppose that G has a 4-leaf basic g -network root N . For any cycle $C = [A_1 A_2 \dots A_k, A_1]$ of $\mathcal{C}'(G)$, $N[A_i]$ is either an invisible star or a visible star, and it is an invisible star iff $|A_{i-1} A_i| = |A_i A_{i+1}| = 1$.

Proof: C is not a 4-cycle in any subgraph of $\mathcal{C}_3(G)$ because $C \in \mathcal{C}'(G)$. By Lemma 4, any root N of G contains a cycle c such that $l(c) = l(C)$.

The first case of the proof of Lemma 4 can not happen because in this case, the corresponding maximal cliques graph is either a triangle or a 4-cycle having all edges of weight 2, so can not be in $\mathcal{C}'(G)$.

For the second case, $N[A_i]$ is either a star or an invisible star for any i . Remark that if an invisible star has two common vertices with another star or invisible star, then a 4 cycle is created. This cycle intersect with c , a contradiction. Hence an invisible star here has at most one common vertex with other stars. In other words, if $N[A_i]$ is an invisible star then $A_{i-1}A_i, A_iA_{i+1}$ have weight 1. Conversely, suppose that $N[A_i]$ is a visible star, we will prove that either $|A_{i-1}A_i|$ or $|A_iA_{i+1}|$ is greater than 1. If $N[A_{i-1}]$ (resp. $N[A_{i+1}]$) is also visible star, then A_iA_{i-1} (resp. A_iA_{i+1}) has weight 2. If both $N[A_{i-1}]$ and $N[A_{i+1}]$ are invisible stars, then by Lemma 3 (ii), the middle vertex of $N[A_i]$ is not adjacent to any other inner vertex not on c . The later implies that the visible star $N[A_i]$ consists of only a middle vertex and a leaf, i.e. it is included in $N[A_{i-1}]$ and $N[A_{i+1}]$, a contradiction. So, $N[A_i]$ is an invisible star if and only if the edges $A_{i-1}A_i, A_iA_{i+1}$ have weight 1. \square

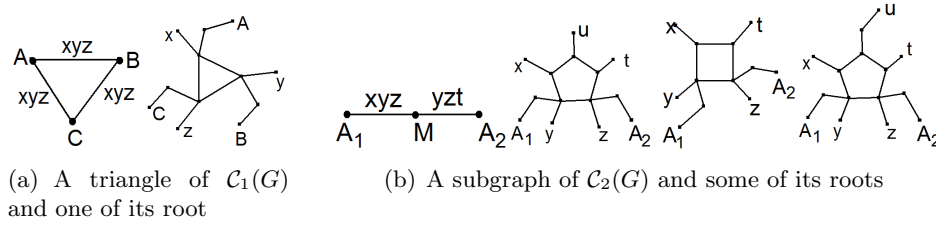


Figure 9: Illustration of Lemma 6

Lemma 6 Suppose that G has a 4-leaf basic g -network root N , then:

1. For each triangle $A_1A_2A_3 \in C_1(G)$, $N[A_1], N[A_2], N[A_3]$ are visible quasi stars sharing the same triangle.

2. For any star $S = MA_1 \dots A_k$ of $C_2(G)$ where M is the middle node and $k \geq 2$, each $N[A_i]$ is a visible star, $N[M]$ is a cycle having the label set equals to $l(M)$, and:

- either $|M| = 5$, then $N[M]$ is an N_5 .
- or $|M| = 4$, then:

If there is not any edge of weight 2 in $MC(G)$ incident to M , then $N[M]$ is an N_4 .

Otherwise, $N[M]$ is a N'_5 and $k = 2$. Let MM' be an edge of weight 2 in $MC(G)$. Then, if $l(MM')$ is disjoint with $l(MA_1) \cap l(MA_2)$ then $N[M']$ is the invisible star having the only invisible vertex on the cycle of $N[M]$ as its middle vertex. Otherwise, $N[M']$ is a visible star.

3. For each subgraph S of $C_3(G)$, $N[S]$ contains a 6-cycle c without invisible vertex such that $l(c) = l(S)$. Moreover, for any node A of S which has a neighbour node in $MC(G)$ not in S , $N[A]$ is a visible star.

Proof: 1. This result is the Claim (iii) of Observation 2.

2. For any $i = 1, \dots, k$, by Claim (v) of Observation 2 we deduce that either both $N[M]$ and $N[A_i]$ are visible quasi stars, or exactly one of them is an N_4 or N_5 or N'_5 . If $N[M]$ is a visible quasi star, then there can not be 2 edges having weight 3 incident to M , except when MA_i is in a triangle of $C_1(G)$ which is a contradiction with the definition of $C_2(G)$. So, $N[M]$ is either an N_4 or N_5 or N'_5 . Remark that in these cases the label set of the cycle in $N[M]$ is equal to $l(M)$. So:

- If $|M| = 5$ then $N[M]$ is an N_5 .

- If $|M| = 4$ then $N[M]$ is either an N_4 or N'_5 .

Remark that if $N[M]$ is an N_4 , then all edges of M have weight 3. So, if there is an edge MM' of weight 2, then $N[M]$ must be a N'_5 (the last cycle in Figure 9(b)). In this figure, we can see that only the two stars having $p(y)$ and $p(z)$ as middle vertices can have 3 common leaves with $N[M]$. Therefore, if $l(MM')$ is disjoint with $l(MA_1) \cap l(MA_2)$, then $l(MM') = \{x, t\}$, and $N[M']$ is the invisible star having the invisible vertex of $N[M]$ as its middle vertex.

If there is not any edge of weight 2 incident to M , then $N[M]$ can be either an N_4 or N'_5 . However, with the constraints that we impose on N (Figure 3(d)), we choose $N[M]$ as an N_4 .

3. In the proof of Lemma 4, we proved that if S is a C_3 , then $N[S]$ contains a 6-cycle c without invisible vertex such that $l(c) = l(S)$. In a $S(c)$, there are 8 networks of the family \mathcal{F} consisting of 2 subnetworks N_6 and 6 visible stars. Observing that if a maximal clique A corresponds to an N_6 then it can not have a neighbour node not in S , otherwise another cycle intersecting with c will be created. So if a node A of S has a neighbour not in S , then it $N[A]$ must be a visible star. \square

Lemmas 5 and 6 are constructive. So it is possible to construct the roots of each subgraph $S \in \mathcal{C}_e(G)$ if we can determine the label of the middle vertex of each visible (quasi) star in the root. This is also easily done with the following remark.

Remark 1 (Middle vertex) *Let N be a 4-leaf basic g -network root of G . Let A be a node of $MC(G)$ such that $N[A]$ is a visible (quasi) star. Let B be a neighbour of A in $MC(G)$ such that $|AB| \geq 2$ and $N[B]$ is not an N_6 . Then the label of the middle vertex of $N[A]$ is contained in $l(AB)$.*

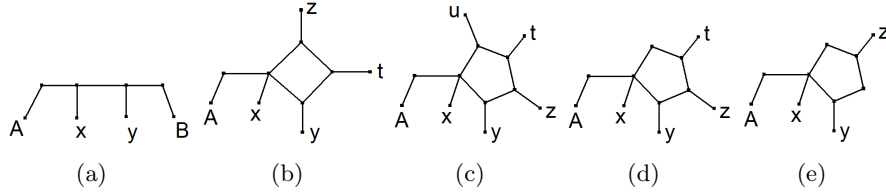


Figure 10: The leaf of the middle vertex of $N[A]$ is a leaf of $N[B]$

This remark can be easily proved by checking all the possible configurations of $N[B]$. Indeed, remark that $N[B]$ is not an invisible (quasi) star because otherwise $|AB| = 1$. For any other configurations of $N[B]$, the leaf of the middle vertex of $N[A]$ is always a leaf of $N[B]$ (see Figure 10).

So, the choice of the label of the middle vertex of $N[A]$ is unique except in some case when A has at most one such neighbour node. For the later case, let B be the only neighbour of A such that $|AB| \geq 2$. Then it depends on the configuration of $N[B]$ that we can deduce easily the label of the middle vertex of $N[A]$. If B does not have neither a further neighbour C such that $|BC| \geq 2$, then the choice of the label of the middle vertex of $N[A]$ does not influence the construction of the remaining part of N . Remark that for any $S \in \mathcal{C}_e(G)$, except the label of the middle vertices of the visible (quasi) stars, the other part of the roots of S is uniquely determined by Lemmas 5, 6. Hence, with the above remark, we can consider all roots of S as only one root, because they have the same configuration and by replacing one by another, we do not change its 4-leaf power as well as the properties of disjoint cycles. With this convention, we have the following result.

Corollary 4 *Each subgraph of $\mathcal{C}_e(G)$ has a unique root.*

By Lemmas 5, 6, and by the condition of vertex-disjoint cycles in the roots of G , we deduce that all subgraphs of $\mathcal{C}'(G)$, $\mathcal{C}_1(G)$, $\mathcal{C}_3(G)$ must be node disjoint and not contain any middle node of any subgraph of $\mathcal{C}_2(G)$.

4.4.2 Other cycles when G is biconnected

The other cycles in the roots of G when G is biconnected consist of all small cycles which are not detected in the previous section. We will use the subgraphs C_4, \dots, C_9 of $MC(G)$ to recognize them. They are not *easy* cycle because each such subgraph can have several roots containing different cycles, and we must consider furthermore conditions to determine which cycles are contained in a 4-leaf basic g-network root of G .

Observation 3 *For $i = 4, \dots, 9$, every 4-leaf basic g-network root of C_i contains a small cycle, and any subgraph smaller than C_i does not have this property.*

Indeed, using Observation 1 and by considering all the possible configurations, we find for each C_i a finite list of its possible roots. They are the networks in Figures 11, ..., 16. *Each root has exactly one cycle.* Note that C_9 can actually have some more roots but by the constraints imposed on the networks, its root have only 2 forms $R_{9,1}, R_{9,2}$. However, not all networks listed in Figures 11, ..., 16 are always roots of the corresponding C_i . We must check furthermore the size of each maximal clique. For example, with C_4 , if the maximal clique D contains more than 3 vertices, then $R_{4,3}$ can not be its root because in this network D corresponds to the maximal clique having 3 vertices x, y, z .

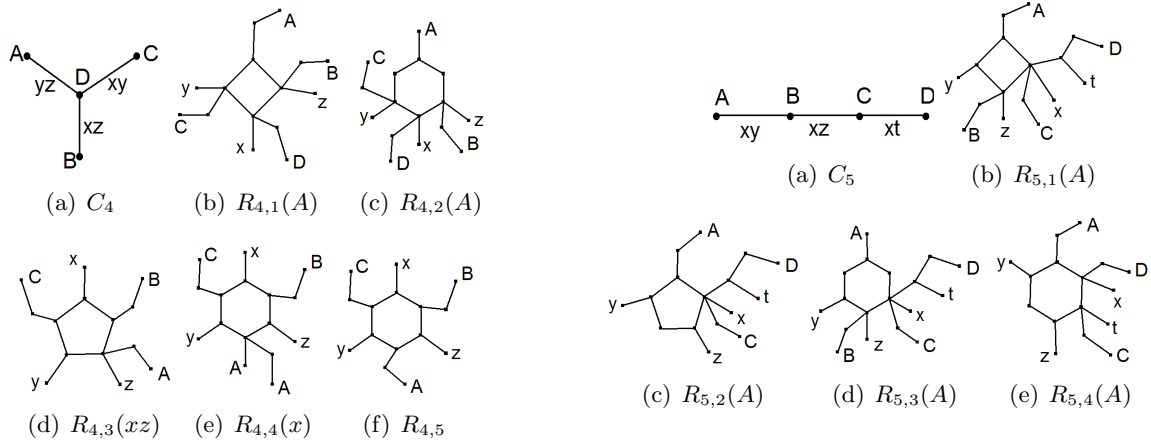


Figure 11: C_4 and its 4-leaf basic g-network roots

Figure 12: C_5 and its 4-leaf basic g-network roots

Example 2 (Roots of C_i) C_5 has 4 forms or roots: $R_{5,1}, R_{5,2}, R_{5,3}, R_{5,4}$. Each form can have several derivations: for the form $R_{5,1}$, we have $R_{5,1}(A)$ as in Figure 12(b) and also $R_{5,1}(D)$ by permuting A and D , B and C . In general, the list of roots of each C_i is generated up to permutation with respect to symmetry.

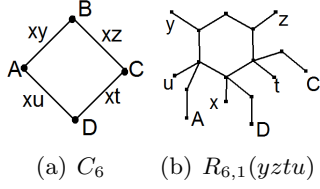


Figure 13: C_6 and its 4-leaf basic g-network roots

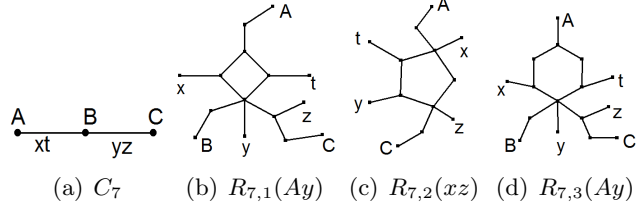


Figure 14: C_7 and its 4-leaf basic g-network roots

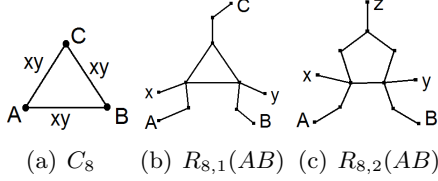


Figure 15: C_8 and its 4-leaf basic g-network roots

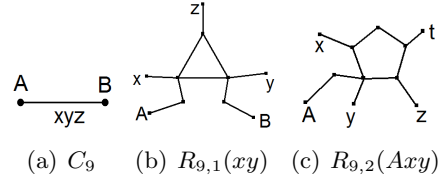
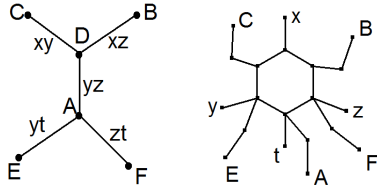


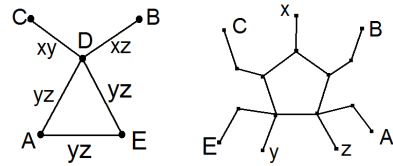
Figure 16: C_9 and its 4-leaf basic g-network roots

These subgraphs in $MC(G)$ are not necessarily node-disjoint, and they are not necessarily node-disjoint with the subgraphs of $\mathcal{C}_e(G)$. Moreover, two different subgraphs can contain same cycle in their roots. We give here 2 examples.

Example 1: the subgraph in the following figure consists of two subgraphs C_4 and its root contains only one cycle.



Example 2: the subgraph in the following figure consists of a subgraph C_4 and a subgraph C_8 . Its root contains only one cycle.



By considering all possible cases, we obtain a list of subgraphs in Figure 17 such that each one consists of 2 intersecting C_i, C_j but its root contains exactly one cycle. Denote by $\mathcal{C}_i(G)$ the set of all subgraphs in $MC(G)$ isomorphic to C_i and not contained in any subgraphs of $\mathcal{C}_e(G)$. Moreover, to avoid the redundancy of the cycles that they infer, for each subgraph isomorphic to one in Figure 17, we count only one of the two C_i, C_j into the corresponding list. In the figure, the chosen one is in bold.

Let $\mathcal{C}_o(G) = \cup \mathcal{C}_i(G)$ for $i = 4, \dots, 9$ (o for other), then we have the following remark. *Suppose that G has a 4-leaf basic g-network root N . Let S_1, S_2 be two distinct subgraphs of $\mathcal{C}_e(G) \cup \mathcal{C}_o(G)$, then the cycles in $N[S_1]$ and $N[S_2]$ are distinct.*

Lemma 7 *Suppose that G is biconnected and has a 4-leaf basic g-network root. Then there is a 4-leaf basic g-network root N of G such that any cycle of N is contained in a root of a subgraph of $\mathcal{C}_e(G) \cup \mathcal{C}_o(G)$.*

Proof: Let c be a cycle of N . We consider each possible configuration of c .

contained in a root of a subgraph of $\mathcal{C}_8(G)$. Otherwise, we can replace it by the network in Figure 18(a) which does not contain any cycle and the resulting network is still a root of G .

If c is a 4-cycle with one invisible vertex: If c is contained in a network as $R_{4,1}$ or $R_{5,1}$ or $R_{7,1}$, then c is contained in a root of a subgraph of $\mathcal{C}_4(G)$ or $\mathcal{C}_5(G)$ or $\mathcal{C}_7(G)$. Otherwise, we can replace it by the network in Figure 18(b) right.

If c is a 4-cycle with two invisible vertices (Figure 18(c) left), then $MC(G)$ consists of only nodes AB connected by an edge of weight 2. So, N in the Figure 18(c) right is a root of G and it does not contain any cycle.

If c is a 5-cycle with one invisible vertex: If c is contained in a network corresponding to a root of \mathcal{C}_2 or as $R_{5,2}$ or $R_{7,2}$, then c is contained in a root of a subgraph of $\mathcal{C}_2(G)$ or $\mathcal{C}_7(G)$. Otherwise, we can replace it by the network in Figure 18(d) right.

If c is a 5-cycle with two invisible vertices: If c is contained in a network as $R_{4,3}$ or $R_{5,2}$ or $R_{8,2}$, then c is contained in a root of a subgraph of $\mathcal{C}_4(G)$ or $\mathcal{C}_5(G)$ or $\mathcal{C}_8(G)$. Otherwise, we can replace it by the network in Figure 18(e) right.

If c is a 6-cycle with two invisible vertices of distance 2: If c is contained in a network as $R_{4,2}$ or $R_{4,4}$ or $R_{5,3}$ or $R_{5,4}$ or $R_{7,3}$, then c is contained in a root of a subgraph of $\mathcal{C}_4(G)$ or $\mathcal{C}_5(G)$ or $\mathcal{C}_7(G)$. Otherwise, we can replace it by the network in Figure 18(f) right.

Finally, if c is a 6-cycle with 3 invisible vertices, then c is contained in a root of a subgraph of $\mathcal{C}_4(G)$. \square

So, if G is biconnected, the sets $\mathcal{C}_e(G)$ and $\mathcal{C}_o(G)$ capture all the cycles that appear in a root of G , and different cycle is contained in a root of a different subgraph. There may exist some roots of G which contain other cycles but we can always replace them by the subnetworks without cycles as in Figure 18.

Further conditions for the roots of \mathcal{C}_i , $i = 4, 5, \dots, 9$

With a given $S \in \mathcal{C}_o(G)$, not every root of S can always appear in some basic g-network root of G , if there is any. For example, let $S \in \mathcal{C}_4(G)$ as in Figure 11(a). If in $MC(G)$, A has a neighbour node not in S , then by Lemma 3 (iii), $N[A]$ can not be an invisible (quasi) star. So, the root $R_{4,1}(A)$ of S can not be presented in any 4-leaf basic g-network root of G . A further example is the following.

Example 3 Let $MC(G)$ be the graph in Figure 19 where A is a maximal clique of G consisting of $\{m, k, u, s\}$. S consists of M, A is a subgraph of $\mathcal{C}_9(G)$. Let us consider the two roots R_1, R_2 of S . R_1 is an induced subgraph of a 4-leaf basic g-network root of G . However, if we choose R_2 as a root of S to reconstruct a root of G , we obtain a network having intersecting cycles. So, R_2 will be not chosen to construct 4-leaf basic g-network roots of G .

Before presenting the way to choose the appropriate roots for each subgraph of $\mathcal{C}_o(G)$, we show some properties that these roots must have.

For any $S \in \mathcal{C}_o(G)$, we denote by $\text{ext}(S)$ the induced subgraph of $MC(G)$ consisting of S and all nodes of $MC(G)$ having distance at most 2 to at least a node of S . For example in Figure 19, if S is the edge MA , then $\text{ext}(S)$ is the path $[M, A, B, C]$.

Lemma 8 Let A be a node of a subgraph $S \in \mathcal{C}_o(G)$. Let us consider the nodes around S in $\text{ext}(S)$. For any 4-leaf basic g-network root N of G :

- (i) If A has a neighbour node not in S , then $N[A]$ is not an invisible (quasi) star.

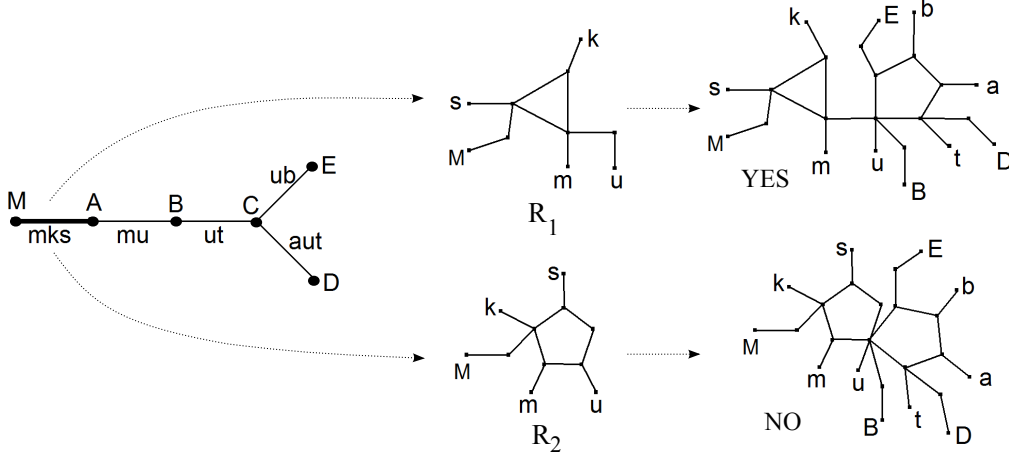


Figure 19: An example of $r(S)$

(ii) Suppose that there are two nodes $B, C \notin S$ such that B is a neighbour of A , and C is a neighbour of B but not of A :

- If $S \notin \mathcal{C}_9(G)$, then $N[A]$ is a visible star.
- If $S \in \mathcal{C}_9(G)$, then $N[A]$ is either a visible (quasi) star or N'_5 . Moreover, if $N[A]$ is an N'_5 , then $N[B]$ is a visible star.

Proof: (i) This claim is deduced from Lemma 3 (iii).

(ii) By Claim (i), $N[A]$ can not be an invisible (quasi) star because otherwise it can not have the neighbour B .

- Suppose that $S \notin \mathcal{C}_9(G)$: By observing the lists of roots of each C_i where $i = 4, \dots, 8$, we see that $N[A]$ is either a visible star, or N'_5 or N''_5 or N_6 .

The only configuration of $N[S]$ such that $N[A]$ is an N'_5 is $R_{7,2}$ (Figure 14(c)), i.e. $S \in \mathcal{C}_7(G)$. Then $N[B]$ is necessarily a visible star having the middle vertex on the cycle of $N[A]$, and B must have 3 common vertices with A . So, (A, B) corresponds to a subgraph S_1 in either $\mathcal{C}_9(G)$ or $\mathcal{C}_2(G)$. It means that $S \cup S_1$ is isomorphic to the subgraph in either Figure 17(h) or 17(i). However, by definition, the cycle of $N[A]$ is implied from S_1 , not from S , and S is not chosen to be in $\mathcal{C}_7(G)$, a contradiction.

If $N[A]$ is N''_5 , then the possible configurations of $N[S]$ are $R_{4,3}, R_{5,2}, R_{8,2}$. Moreover, $N[S]$ can not be $R_{8,2}$ because in this network, all 2 possible neighbours of the maximal cliques corresponding to N''_5 are in S . It means that A can not have such a neighbour B . So, $N[S]$ has form of either $R_{4,3}$ or $R_{5,2}$, i.e. S is either in $\mathcal{C}_4(G)$ or in $\mathcal{C}_5(G)$. $N[B]$ can not be an invisible (quasi) star because otherwise it can not have such a neighbour C (by Lemma 3 (iii)). $N[B]$ can not contain a cycle because otherwise its cycle intersects with the cycle of $N[A]$. So, $N[B]$ must be a visible star having the middle vertex on $N[A]$. However, that will create a triangle ABX in $MC(G)$, where X is the neighbour of A in S such that $N[X]$ is a visible star having the middle vertex on $N[A]$. Let $S_1 = (ABX)$, so S_1 is a subgraph of $\mathcal{C}_8(G)$. It means that $S \cup S_1$ is isomorphic to the subgraph in Figure 17(e) or 17(g). However, by definition, in this case S is not chosen to be in $\mathcal{C}_4(G)$ or $\mathcal{C}_5(G)$, and the corresponding cycle in $N[A]$ is implied from S_1 , not from S .

Similarly, if $N[A]$ is N_6 , by analysing the lists of roots as above, we deduce a contradiction.

Hence, $N[A]$ is a visible star.

- Suppose that $S \in \mathcal{C}_9(G)$: So $N[S]$ has form of either $R_{9,1}$ or $R_{9,2}$ where $N[A]$ is either a visible (quasi) star or isomorphic to N'_5 . In the latter case, if $N[B]$ contains a 4, 5, 6-cycle, it will intersect with the cycle of $N[A]$. If $N[B]$ is an invisible (quasi) star then its middle vertex must be on the cycle of $N[A]$, and B does not have the neighbour C , a contradiction. If $N[B]$ is a visible quasi star then its triangle intersect with the cycle of $N[A]$. Hence, $N[B]$ must be a visible star. \square

By observing the lists of roots of $\mathcal{C}_o(G)$, we deduce the following.

Observation 4 (i) For $i = 4, 6, 8, 9$, $\forall S \in \mathcal{C}_i(G)$, every root of S contains a cycle c such that $l(S) \subseteq l(c)$.

(ii) $\forall S \in \mathcal{C}_5(G)$, denote by $lc(S)$ the only common vertex of the 4 maximal cliques of S , then every root of S contains a cycle c such that $lc(S) \in l(c)$.

Example 4 (i) The subgraph C_4 in Figure 11(a) has several roots, but the cycles in all of its roots always have the label set contains $\{x, y, z\}$, i.e. equals to $l(S)$.

(ii) The subgraph C_5 in Figure 12(a) has several roots, but the cycles in all of its roots always have the label set containing x , which is the common vertex of the 4 maximal cliques of this subgraph.

The following definition shows how to determine the appropriate roots of each subgraph of $\mathcal{C}_o(G)$.

Definition 4 Let $S \in \mathcal{C}_o(G)$, we define $\mathbf{r}(S)$ as the set of roots r of S such that:

1. The cycle c in r is vertex-disjoint with the cycle in the root of every subgraph of $\mathcal{C}_e(G)$. Moreover, $l(c)$ is disjoint with $l(S')$ for any $S' \in \mathcal{C}_i(G)$, $i = 4, 6, 8, 9$ and it is disjoint with $lc(S')$ for any $S' \in \mathcal{C}_5(G)$.

2. r is the induced subgraph of a 4-leaf basic g -network r' where r' is a root of $\text{ext}(S)$ which satisfies the properties in Lemma 8.

The following properties of $\mathbf{r}(S)$ can be deduced from the above definition.

Lemma 9 Let $S \in \mathcal{C}_o(G)$, and let r_0, r_1 be two distinct networks of $\mathbf{r}(S)$. For $i = 0, 1$, let r'_i be a root of $\text{ext}(S)$ which satisfies the properties in Lemma 8 such that r_i is the induced subgraph of r'_i on S . Let A be a node of S which has a neighbour B and B has a neighbour C not adjacent with A such that $B, C \notin S$, then:

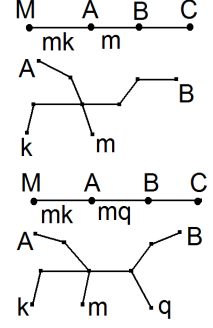
1. $r'_0[A \cap B]$ and $r'_1[A \cap B]$ have the same configuration.
2. Denote by $r'_{0,1}$ the network obtained by overlaying the common part of $r'_0[S]$ and $r'_1[B \cup C]$ on $A \cap B$. Then the cycles in $r'_{0,1}$ are vertex-disjoint.

Proof: Remark that $d_{MC(G)}(A, B) = 1$, $d_{MC(G)}(A, C) = 2$, so $B, C \in \text{ext}(S)$.

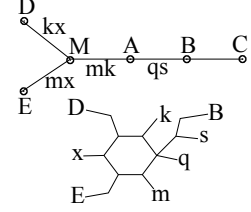
1. Let M be a neighbour of A in S , and let r' be a certain 4-leaf basic g -network root of $\text{ext}(S)$ which satisfies the properties in Lemma 8. We will prove that $r'[A \cap B]$ has only one configuration by considering all possible cases:

- (i) If $S \notin \mathcal{C}_9(G)$, let $l(AM) = \{m, k\}$. By Lemma 8 (ii), $r'[A]$ is a visible star.

(i)-a If $|AB| = 1$, let $l(AB) = \{m\}$, then $r'[A]$ has $p(m)$ as its middle vertex, and $r'[B]$ must be an invisible (quasi) star (according to Observation 2 (iv)). The configuration of $r'[A \cap B]$ is as the figure on the right.



(i)-b If $|AB| = 2$ and $l(AM) \cap l(AB) = \{m\}$, let $l(AB) = \{m, q\}$, then due to Remark 1, $p(m)$ must be the middle vertex of $r'[A]$. We have the configuration as the figure on the right.



(i)-c If $|AB| = 2$ and $l(AM) \cap l(AB) = \emptyset$, then $S' = (M, A, B)$ is a subgraph C_7 .

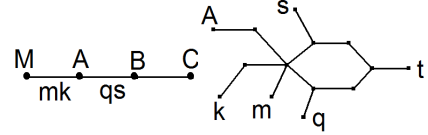
If $S \cup S'$ has the configuration $C_4 + C_7$ as in Figure 17(d) where S is a C_4 , then $r'[M]$ is a N_6 , $r'[A]$ is a visible star. We have the configuration as the figure in the right.

Otherwise, the cycles in the root of S and S' are distinct. S' can have 3 forms of roots: $R_{7,1}$, $R_{7,2}$, $R_{7,3}$.

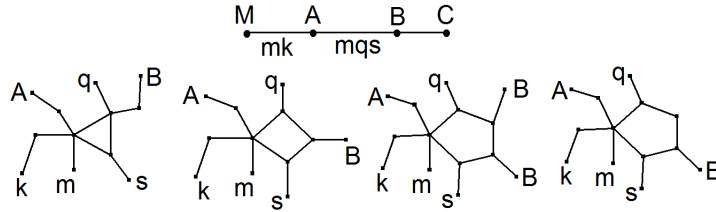
- If $r'[M, A, B]$ has form $R_{7,1}$, and $r'[M]$ is an invisible star, then the cycle in the root of S and S' is the same, a contradiction. So in this case $r'[B]$ is an invisible star, i.e. B can not have a neighbour C , a contradiction.

- If $r'[M, A, B]$ has form $R_{7,2}$, then all the four vertices $p(m)$, $p(k)$, $p(q)$, $p(s)$ are contained in the cycle of this root. However, by observing the lists of roots of each C_i , we see that either $p(m)$ or $p(k)$ must be contained in the cycle of every root of S . So, these two cycles intersect, contradicting r' is a g-network.

- Hence, $r'[M, A, B]$ has form $R_{7,3}$ where $r'[B]$ is an N_6 as in the figure on the right and the middle vertex of $r'[A]$ is on the cycle of $r'[B]$.

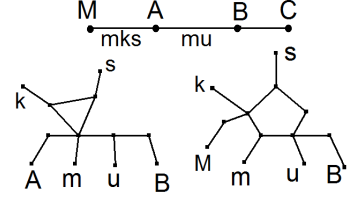


(i)-d If $|AB| = 3$, let $l(AB) = \{m, q, s\}$, then according to Observation 2 (ii), $r'[B]$ is either a visible quasi star or N_4 or N_5 or N'_5 . By Remark 1, $p(m)$ is the middle vertex of $r'[A]$. We have the configuration in the following figure.

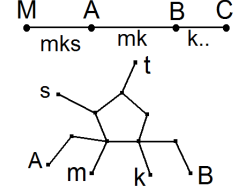


(ii) If $S \in \mathcal{C}_9(G)$, let $l(AM) = \{m, k, s\}$, then $|AB| = 2$ and $l(AB) \cap l(AM) \neq \emptyset$. The roots of S have two forms: $R_{9,1}$ and $R_{9,2}$. There are 2 following cases.

(ii)-a $|l(AM) \cap l(AB)| = 1$, let $l(AB) = \{m, u\}$. $r'[B]$ must be a visible (quasi) star having $p(u)$ as its middle vertex. If $r'[S]$ has form $R_{9,1}$ where $r'[A]$ and $r'[M]$ are two visible quasi star sharing the same triangle, then by Remark 1, the middle vertex of $r'[A]$ is $p(m)$. If $r'[S]$ has form $R_{9,2}$ then $r'[A]$ is an N'_5 and $r'[B]$ is a visible star. Moreover, if $r'[B, C]$ contains a cycle passing $p(u)$, then $N[S]$ has form $R_{9,1}$ because otherwise the cycle in $r'[B, C]$ will intersect with the cycle in $r'[S]$.



(ii)-b $|l(AM) \cap l(AB)| = 2$, let $l(AB) = \{m, k\}$. Suppose that $r'[S]$ has form $R_{9,1}$, then for any configuration of $r'[B]$, $|l(MA) \cap l(AB)| \leq 1$, a contradiction. So, $r'[S]$ has form $R_{9,2}$ where $r'[A], r'[B]$ are two visible stars having either $p(m)$ or $p(k)$ as their middle vertices. Moreover, if $l(AB) \cap l(BC) = \{k\}$, then $r'[B]$ has $p(k)$ as its middle vertex, and $r'[A]$ has $p(m)$ as its middle vertex.



We see that in each case the possible configurations of r' on $A \cap B$ are unique even in the case that it has several possible configurations on M, A, B, C as in the cases (i)-d, (ii)-a. In other words, $r'_0[A \cap B]$ and $r'_1[A \cap B]$ have the same configuration.

2. Due to Claim 1, we can combine $r'_0[S]$ with $r'_1[B, C]$ by overlaying their common part on $A \cap B$ to obtain the network $r'_{0,1}$. So we have $r'_{0,1}[S] = r'_0[S]$ and $r'_{0,1}[B, C] = r'_1[B, C]$. Suppose that $r'_{0,1}$ has two intersecting cycles c_0, c_1 . Then c_0 is contained in $r'_{0,1}[S]$ and c_1 is contained in $r'_{0,1}[B, C]$. This claim will also be proved by considering all possible cases that are analysed above.

(i) If $S \notin \mathcal{C}_9(G)$: In this case $r'_{0,1}[A]$ is a visible star. So c_0 is contained in $r'_{0,1}[M]$, and it passes the middle vertex of $r'_{0,1}[A]$. In order that c_0 and c_1 intersect, the cycle c_1 must be contained in $r'_{0,1}[B]$ and c_0, c_1 intersect at the middle vertex of $r'_{0,1}[A]$.

(i)-a As proved above, $r'_{0,1}[B]$ must be an invisible (quasi) star. Then in order that $r'_{0,1}[B]$ contains a cycle, it must be an invisible quasi star and its triangle contains $p(m)$. However, the other leaf of this triangle is also contained in A . It means that A, B have more than one common vertices, a contradiction.

(i)-b We consider furthermore the node C to determine the possible forms of $r'[B]$ where r' is a certain 4-leaf basic g-network root of G which satisfies the properties in Lemma 8. In this case, $r'[B]$ is either a visible (quasi) star, N'_5 , N''_5 or N_6 . It can be checked that each possible configuration of BC brings about a unique configuration of $r'[B]$. So $r'_0[B]$ and $r'_1[B]$ are the same. Since r'_0 is a g-network, $r'_0[M]$ and $r'_0[B]$ do not contain intersecting cycles. Hence, the cycles in $r'_{0,1}[M] = (r'_0[M])$ and $r'_{0,1}[B] = (r'_1[B] = r'_0[B])$ do not intersect.

(i)-c For a certain r' , it can be seen that $r'[B]$ has only one configuration which is N_6 and this cycle passes the middle vertex of $r'[A]$. Hence, the cycles in $r'_0[B]$ and $r'_1[B]$ are the same. Similarly with the above case, we deduce that the cycles in $r'_{0,1}[M]$ and $r'_{0,1}[B]$ do not intersect.

(i)-d It can be seen that in all possible configurations, $r'_0[B]$ and $r'_1[B]$ contain a cycle passing $p(m)$. If $r'_{0,1}[M]$ and $r'_{0,1}[B]$ have $p(m)$ as a common vertex, then $r'_0[M] = (r'_{0,1}[M])$ and $r'_0[B]$ also have $p(m)$ as a common vertex. The later is a contradiction because r'_0 is a g-network.

(ii) If $S \in \mathcal{C}_9(G)$:

(ii)-a Suppose that r'_0 corresponds to the first configuration and r'_1 corresponds to the second configuration of the case (ii)-a in Claim 1. Then $r'_{0,1}[A]$ is a visible quasi star having $p(m)$ as its middle vertex and $r'_{0,1}[B]$ is a visible star having $p(u)$ as its middle vertex. It means that the cycle c_0 is the triangle of $r'_{0,1}[A]$, and c_1 must be contained in $r'_{0,1}[C]$. However, these two cycles

do not intersect.

Suppose that r'_0 is the second configuration and r'_1 is the first configuration of the case (ii)-a in Claim 1. Then c_0 is contained in cycle N'_5 of $r'_{0,1}[A]$. We know that $r'_{0,1}[B, C]$ is a visible (quasi) star. In order that $r'_{0,1}[B, C]$ contains a cycle intersecting with c_0 , either $r'_{0,1}[B]$ is a visible quasi star or $r'_{0,1}[C]$ contains a cycle passing the middle vertex of $r'_{0,1}[B]$. In the first case, we deduce that $r'_1[A]$ and $r'_1[B]$ have 3 common leaves, a contradiction because $|AB| = 2$. In the second case, we deduce that C is contained in a subgraph S_1 of $\mathcal{C}_i(G)$ where $i \in \{4, 5, 6, 8, 9\}$. So, the label of the middle vertex of $r'_1[B]$ intersects with either $l(S_1)$ if $i \in \{4, 6, 8, 9\}$ or $lc(S_1)$ if $i = 5$. The later shows that r'_1 does not satisfy the property 1 of Definition 4, i.e $r'_1 \notin r(S)$, a contradiction.

(ii)-b $r'[A, B]$ has a unique configuration as indicated in the case (ii)-b of Claim 1. In this configuration $r'[A]$ contains the cycle c_0 and $r'[B]$ is a visible star. So, c_0 is contained in $r'_{0,1}[A]$ and c_1 is contained in $r'_{0,1}[C]$. Moreover, $r'_0[A] = r'_1[A]$. If c_0 intersects with c_0 , then the cycle in $r'_1[C] = (r'_{0,1}[C])$ also intersects with the cycle in $r'_1[A] = (r'_0[A] = r'_{0,1}[A])$. The later is a contradiction because r'_1 is a g-network. \square

Theorem 3 *Let G be a biconnected graph which has a 4-leaf basic g-network root N_0 , and $S \in \mathcal{C}_o(G)$.*

1. *Let $r_0 = N_0[S]$, then $r_0 \in r(S)$*
2. *For any other network $r_1 \in r(S)$ different from r_0 , we can replace r_0 in N_0 by r_1 and the obtained network N_1 is also a 4-leaf basic g-network root of G .*

Proof: (i) It is obvious that r_0 is a root of S . We have to show that r_0 satisfies the 2 conditions in Definition 4. Let c be the cycle in r_0 . The first condition: we know that the cycle in the root of each subgraph in $\mathcal{C}_e(G)$ is contained in N_0 , then c can not intersect with any of them. For any $S_1 \in \mathcal{C}_o(G)$ different from S , $N_0[S_1]$ contains a cycle different from c . So, $l(c)$ can not intersect with the leaf set $l(S_1)$ if $S_1 \in \mathcal{C}_i(G)$ where $i = 4, 6, 8, 9$ or with $lc(S_1)$ if $S_1 \in \mathcal{C}_5(G)$. The second condition is obviously satisfied because $r'_0 = N_0[ext(S)]$ is a root of $ext(S)$ and it obviously has the properties in Lemma 8 because N_0 is a 4-leaf basic g-network root of G . Moreover, r_0 is an induced subgraph of r'_0 on S . Therefore, $r_0 \in r(S)$.

(ii) Let r_1 be another root of $r(S)$ different from r_0 (Figure 4), so there is basic g-network r'_1 which is a root of $ext(S)$ and satisfies the properties in Lemma 8 such that $r_1 = r'_1[S]$. We will construct a basic g-network N_1 as follows. Let H_1, \dots, H_m be the connected components of $MC(G)$ obtained after deleting S . Then for each H_i , there are two possible cases.

Case (a): If all nodes of H_i are adjacent to at least a node of S , then denote $N^i = r'_1[H_i]$.

Case (b): Otherwise, let $N^i = N_0[H_i]$.

In case (a), both N^i and r_1 are induced subgraphs of r'_1 , then we can combine N^i with r_1 by overlaying their common part. In case (b), let A be a node of S with has a neighbour B and B has a neighbour C such that $B, C \in H_i$. So the common vertices of H_i and S are contained in $A \cap B$. According to Claim 1 of Lemma 9, $r'_0[A \cap B]$ and $r'_1[A \cap B]$ are the same. Moreover, $r'_0[A \cap B] = N_0[A \cap B] = N^i[A \cap B]$, and $r'_1[A \cap B] = r_1[A \cap B]$. Then $N^i[A \cap B]$ and $r_1[A \cap B]$ are the same. So, we can also combine N^i with r_1 by sticking their common part on $A \cap B$ (Figure 4). Denote the obtained network by N_1^i .

It is easy to see that N_1^i is a 4-leaf root of $S \cup H_i$. Indeed, for case (a), both r_1 and N^i is an induced subgraphs of r'_1 , i.e. $N_1^i = r'_1[S \cup H_i]$. We know that r'_1 is a root of $ext(S)$, so N_1^i is a root of $S \cup H_i$. For case (b), we have $(N_1^i[S])^{4l} = r_1^{4l} = S$, and $(N_1^i[H_i])^{4l} = (N^i)^{4l} = H_i$.

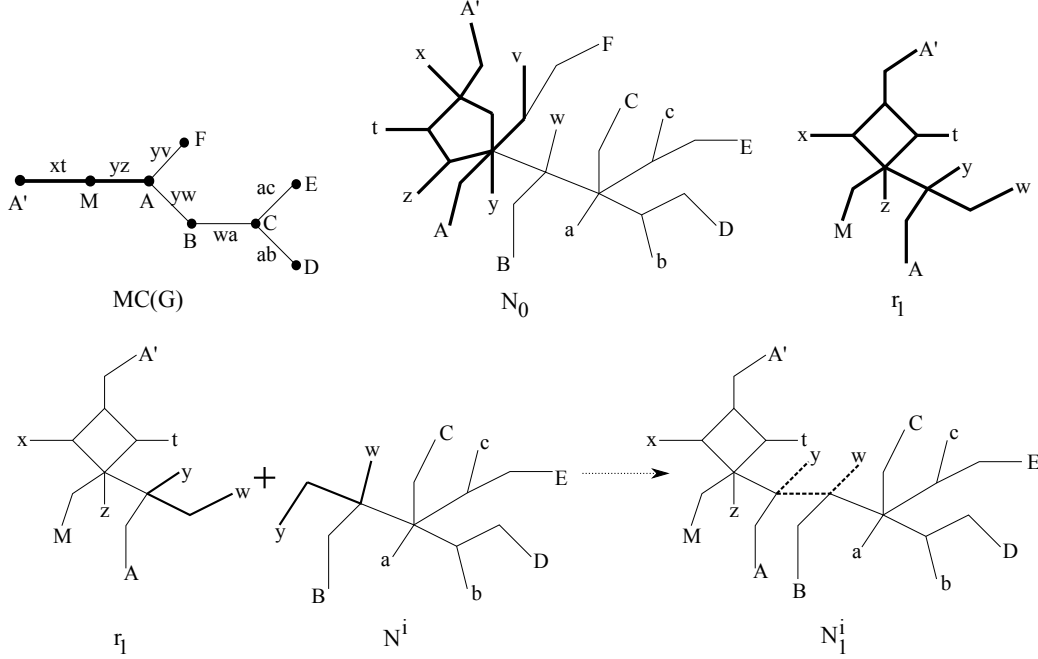


Figure 20: $S = (A', M, A)$. The two subnetworks r_0, r_1 of $r(S)$ are in bold.

Moreover, for any $x \in S \setminus H_i, y \in H_i \setminus S$, $d_{N_0}(x, y)$ and $d_{N_1^i}(x, y)$ are both greater than 4. Hence, $(N_1^i)^{4l} = (N_0[S \cup H_i])^{4l} = S \cup H_i$. Then N_1^i is a 4-leaf root of $S \cup H_i$.

Let N_1 be the network obtained by combining all N^i to r_1 as described above for every H_i . Hence, N_1 is a 4-leaf basic network root of G .

We must show furthermore that N_1 is a g-network, i.e. its cycles are pairwise vertex disjoint. Let c be the cycle of r_1 . All cycles of N_1 different from c are pairwise vertex-disjoint because by construction, they are also the cycles of N_0 and N_0 is a g-network. So, it remains to show that not any cycle in N_1 different from c intersects with c . By Definition 4, c does not intersect with the cycle in the root of any subgraph of $\mathcal{C}_e(G)$. Suppose that there is a cycle c' in N_1 that intersects with c . Because c' does not contained in the root of any subgraph of $\mathcal{C}_e(G)$, it is a small cycle and different from a 6-cycle without invisible vertex. For all possible configurations of c' , it can be checked that c' is contained in $N_1[X]$ where X is a node in $ext(S)$. Moreover, X can not be in S because otherwise r_1 contains two cycles c, c' . So, $X \in ext(S) \setminus S$. Suppose that X is a node of a certain H_i in case (a). Then by construction, N_1^i is an induced subgraph of r_1' . Since r_1' does not contain intersecting cycles, therefore N_1^i can not have intersecting cycles, contradicting the two intersecting cycles c, c' be both in N_1^i . Otherwise X is a node of a certain H_i in case (b). So, there exist two nodes B, C in H^i such that B, C are adjacent and B has a neighbour A in S where either $X = B$ or $X = C$. However, by Lemma 9, $r_1'[S]$ and $r_0'[B, C]$ do not have intersecting cycles. It means that $N_1^i[S] = (r_1'[S])$ and $N_1^i[B, C] = (r_0'[B, C])$ do not have intersecting cycles. In other words, c, c' do not intersect. Hence, N_1 is a basic g-network.

Therefore, N_1 is a 4-leaf basic g-network root of G and N_1 is obtained by replace r_0 in N_0 by r_1 . \square

Using Definition 4 and Lemma 8, one can calculate for each subgraph S of $\mathcal{C}_o(G)$ the root set $r(S)$ basing on the lists of roots in Figures 11, ..., 16 and the analysis of configurations in the proof of Lemma 9. However, Theorem 3 does not hold when G is not biconnected.

4.4.3 Other cycles when G is not biconnected

When G contains several blocks, we consider separately each block and construct 4-leaf basic g-networks roots of each one. However, there are some conditions must be taken into account on the roots of two adjacent blocks.

Lemma 10 *Let $\mathcal{B}_1, \dots, \mathcal{B}_m$ be the blocks of G . Then G is a 4-leaf basic g-network power if and only if each \mathcal{B}_i is a 4-leaf basic g-network power and for any two blocks $\mathcal{B}_i, \mathcal{B}_j$ having a common vertex x , there exist some roots N_i, N_j of $\mathcal{B}_i, \mathcal{B}_j$ such that:*

- *either all the neighbours of $p(x)$ in N_i are invisible or all the neighbours of $p(x)$ in N_j are invisible.*
- *if $p(x)$ is contained in a cycle of N_i (resp. N_j) then it is not contained in any cycle of N_j (resp. N_i).*

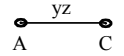
Proof: \Rightarrow Suppose that G has a 4-leaf basic g-network power N . Let $N_i = N[\mathcal{B}_i]$ for any $i = 1, \dots, k$, so each N_i is a 4-leaf basic g-network root of \mathcal{B}_i . Suppose that there is a neighbour C_i of $p(x)$ in N_i and a neighbour C_j of $p(x)$ in N_j which are both visible. Let y_i, y_j be the leaves attached to C_i, C_j . So the distance of y_i, y_j is 4, or $y_i y_j$ are connected in G , which is a contradiction with the fact that $\mathcal{B}_i, \mathcal{B}_j$ are 2 distinct blocks.

The second condition is implied from the fact that N is a g-network and so its cycles are pairwise disjoint.

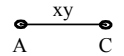
\Leftarrow Suppose that each \mathcal{B}_i has a 4-leaf basic g-network root N_i and the two conditions are satisfied. We construct N by combining all N_i by the same way in [5]: For any cut-vertex x corresponding to a leaf which is a cut-vertex of G , we glue $p(x)$ together and remove all copies of x except one. So N is a 4-leaf root of G and N is a g-network. Indeed, the distance between any 2 leaves in one block does not change, the distance between any 2 leaves of 2 distinct blocks which do not equal the cut-vertex of these blocks are always greater than 4 (from the first condition), and the cycles in N are pairwise disjoint (from the second condition and from the fact that each N_i is a basic g-network). \square

By Lemma 10, the conditions at the cut-vertices of G impose some further cycles in the roots of G apart from those in the roots of the subgraphs in $\mathcal{C}_e(G)$ and $\mathcal{C}_o(G)$. The subgraphs C_{10}, C_{11}, C_{12} in the following allow us to recognize them.

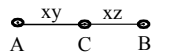
Let $\mathcal{C}_{10}(G)$ be the set of subgraphs of $MC(G)$ which is not contained in any subgraphs of $\mathcal{C}_e(G) \cup \mathcal{C}_o(G)$ and isomorphic to C_{10} in the figure on the right where A contains a cut-vertex of G different from y, z .



Let $\mathcal{C}_{11}(G)$ be the set of subgraphs of $MC(G)$ which is not contained in any subgraphs of $\mathcal{C}_e(G) \cup \mathcal{C}_o(G)$ and isomorphic to C_{11} in figure on the right where x is a cut-vertex of G , and A, C do not have any other neighbours in the same block.



Let $\mathcal{C}_{12}(G)$ be the set of subgraphs of $MC(G)$ which is not contained in any subgraphs of $\mathcal{C}_e(G) \cup \mathcal{C}_o(G)$ and isomorphic to C_{12} in the figure on the right where x is a cut-vertex of G , and A, B, C do not have any other neighbours in the same block.



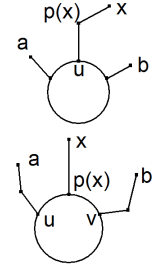
And $\mathcal{C}_c(G) = \mathcal{C}_{10}(G) \cup \mathcal{C}_{11}(G) \cup \mathcal{C}_{12}(G)$ (c for cut-vertex).

The following lemma shows that $\mathcal{C}_c(G)$ allows us to recognize the remaining cycles.

Lemma 11 *Let G be a 4-leaf basic g -network power and let \mathcal{B} be a block of G which is not a clique and contains a cut-vertex x . Suppose that \mathcal{B} has a basic g -network root $N_{\mathcal{B}}$ in which $p(x)$ is adjacent to only invisible vertex. Then $N_{\mathcal{B}}$ has a cycle c which is contained in a root of a subgraph of $\mathcal{C}_e(\mathcal{B}) \cup \mathcal{C}_o(\mathcal{B}) \cup \mathcal{C}_c(\mathcal{B})$ such that $S(c)$ contains x .*

Proof: By Lemma 3 (i), each of invisible vertices adjacent to $p(x)$ is contained in a cycle of $N_{\mathcal{B}}$. Because $N_{\mathcal{B}}$ does not have intersecting cycles, these invisible vertices are contained in an unique cycle.

So, there are only 2 possible cases as in the figures on the right, where u, v are invisible vertices contained in the cycle c . In both two cases, x is contained in $S(c)$. In the first case, x is contained in exactly one maximal clique of \mathcal{B} , corresponding to the invisible star of u . In the second case, x is contained in at most 3 maximal cliques of \mathcal{B} . Two of them correspond to the two invisible stars of u, v , and it may exist the third one corresponding to the cycle c passing $p(x), u, v$. In the latter case, c must be either N_5'' or N_6 because only these networks of \mathcal{F} can have 2 invisible vertices on their cycle.



Suppose that x is not contained in any subgraph of $\mathcal{C}_e(\mathcal{B}) \cup \mathcal{C}_o(\mathcal{B})$, then the cycle c is not contained in any root of $\mathcal{C}_e(\mathcal{B})$ or $\mathcal{C}_o(\mathcal{B})$. So c is one of the cycle in the networks that are replaced by the networks without cycle in the proof of Lemma 7 (Figures 18(a), 18(b), 18(c), 18(d), 18(e), 18(f)). By calculating the corresponding maximal cliques graphs, there are 3 cases:

- x is contained in exactly one maximal clique A of \mathcal{B} , then x is contained in a subgraph of $\mathcal{C}_{10}(\mathcal{B})$.
 - x is contained in exactly two maximal cliques A, C of \mathcal{B} , then x is contained in a subgraph of $\mathcal{C}_{11}(\mathcal{B})$. Indeed, if A, C have neighbours in \mathcal{B} , then by checking the root list we see that the corresponding subgraph does not have any root in which $p(x)$ is adjacent to only invisible vertex, a contradiction.
 - x is contained in exactly three maximal cliques A, B, C of \mathcal{B} , then x is contained in a subgraph of $\mathcal{C}_{12}(\mathcal{B})$. Indeed, if A, B, C have neighbours in \mathcal{B} , then either there is not any root in which $p(x)$ is adjacent to only invisible vertex, or the corresponding subgraph is in $\mathcal{C}_e(G) \cup \mathcal{C}_o(G)$, contradictions.
-

For each subgraph $\mathcal{C}_{10}, \mathcal{C}_{11}, \mathcal{C}_{12}$, we find also a finite list of its corresponding roots (Figures 21, 22, 23). Beside the roots without cycle (Figures 21(f), 22(c), 23(c)), we have also the roots containing cycles for the case that all neighbour vertices of $p(x)$ are invisible (figures 21(b), 21(c), 21(d), 21(e), 22(b), 23(b)). These cycles do not necessarily appear in the roots of one block (because we can replace it by a cycleless network as proved in Lemma 7), but when $p(x)$ needs to be adjacent to only invisible vertices, they necessarily appear.

For any subgraph $S \in \mathcal{C}_c(G)$, we also restrain the choices of roots of S from the full provided lists as in Definition 4. Hence, the obtained root set $r(S)$ has the same properties as the root set of the subgraphs of $\mathcal{C}_o(G)$ in Theorem 3. In fact, only the subgraphs of $\mathcal{C}_{10}(G)$ are concerned here because by definition the subgraphs of $\mathcal{C}_{11}(G)$ and $\mathcal{C}_{12}(G)$ do not have other neighbours in the same blocks.

Denote $\mathcal{S}(G) = \mathcal{C}_e(G) \cup \mathcal{C}_o(G) \cup \mathcal{C}_c(G)$, we have the following lemma.

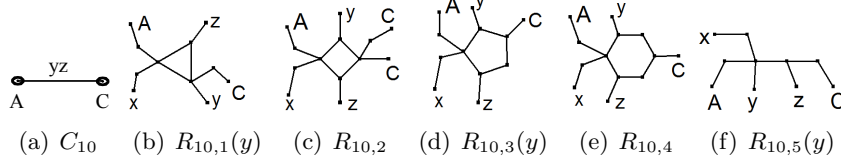


Figure 21: C_{10} and its 4-leaf basic g-network roots, x is a cut-vertex contained in A .

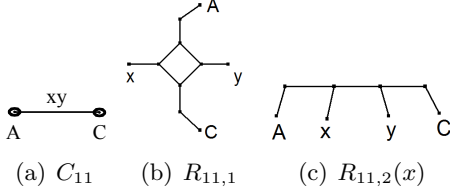


Figure 22: C_{11} and its 4-leaf basic g-network roots, x is a cut-vertex.

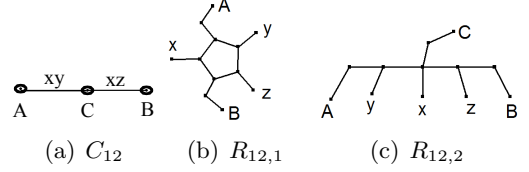


Figure 23: C_{12} and its 4-leaf basic g-network roots, x is a cut-vertex.

Corollary 5 *Suppose that G has at least one 4-leaf basic network root. Then there is a 4-leaf basic network root N of G such that any cycle of N is contained in a root of a subgraph of $\mathcal{S}(G)$.*

Proof: It can be seen that each cycle listed in Figures 18(a), 18(b), 18(c), 18(d), 18(e), 18(f) of Lemma 7 is contained in a root of a subgraph of $\mathcal{C}_c(G)$ in Figures 21, 22, 23. Combining with the proof of Lemma 7, we are done. \square

So, we are going to construct the roots of G in which each cycle is contained in a root of a subgraph of $\mathcal{S}(G)$.

For each block \mathcal{B} which contains a cut-vertex x , we introduce two variables $c_{\mathcal{B}}(x)$ and $i_{\mathcal{B}}(x)$ with the convention that: $c_{\mathcal{B}}(x) = \text{true}$ in a root of \mathcal{B} if $p(x)$ is not contained in any cycle of this root, *false* otherwise; $i_{\mathcal{B}}(x) = \text{true}$ in a root of \mathcal{B} if $p(x)$ is adjacent to only invisible vertices in this root and *false* otherwise. The 2 conditions in Lemma 10 can be rewritten as: in any root of G we have $c_{\mathcal{B}}(x) \vee c_{\mathcal{B}'}(x) = \text{true}$ and $i_{\mathcal{B}}(x) \vee i_{\mathcal{B}'}(x) = \text{true}$ for any two blocks $\mathcal{B}, \mathcal{B}'$ sharing a cut-vertex x . We say that these values are *compatible* in this case.

If \mathcal{B} is a block clique, then by the constraints imposed on the roots of G , we have $c_{\mathcal{B}}(x) = \text{true}$ and $i_{\mathcal{B}}(x) = \text{true}$ because the root of \mathcal{B} is an invisible star. So, there is no constraint imposed by \mathcal{B} on the root of \mathcal{B}' . Hence, we need to consider only the case that neither \mathcal{B} nor \mathcal{B}' is a clique.

If x is not in any subgraph of $\mathcal{S}(\mathcal{B})$, then by Lemma 11, Corollary 5, $p(x)$ is not contained in any cycle and it is adjacent to at least one visible vertex. So in this case, we can impose $c_{\mathcal{B}}(x) = \text{true}$ and $i_{\mathcal{B}}(x) = \text{false}$.

If x is contained in at least a subgraph of $\mathcal{S}(\mathcal{B})$ which has several roots and each one gives a different values for these variables, then we must choose the appropriate one. So, the conditions in Lemma 10 restrain the choices of roots of each subgraph of $\mathcal{S}(G)$ which contains a cut-vertex. For example, the root set of a subgraph S_1 may restrain the choice of roots of a subgraph S_2 which is contained in an adjacent block of the block of S_1 . This restriction may influent the choice of roots of other subgraphs and so on. However, we propose in the following theorem an efficient way to apply these conditions.

Theorem 4 *Given $\mathcal{S}(G)$ and the root set of each subgraph in $\mathcal{S}(G)$, one can find in linear time a root for each subgraph of $\mathcal{S}(G)$, if there is any, such that the conditions in Lemma 10 are satisfied at any cut-vertex of G .*

Proof: Let $BC(G)$ be the *block tree* of G . We choose a certain node R of $BC(G)$ as its root. At the beginning, we impose $root(S) = r(S)$, i.e. the root set of S for any subgraph S of $\mathcal{S}(G)$. We apply the condition in Lemma 10 as follows:

- Begin with the blocks corresponding to the leaves of $BC(G)$: for any subgraph $S \in \mathcal{S}(G)$ of these blocks, we keep $root(S) = r(S)$.

For each block \mathcal{B} of G , denote by $BC_{\mathcal{B}}(G)$ the subtree of $BC(G)$ rooted at \mathcal{B} .

Induction hypothesis: Let \mathcal{B} is the currently considered block, then for any root in $root(S)$ of a subgraph S of \mathcal{B} , there exists at least a root in $root(S')$ for each subgraph S' in any block of $BC_{\mathcal{B}}(G)$ such that the conditions in Lemma 10 are satisfied at any cut-vertex between two adjacent blocks in $BC_{\mathcal{B}}(G)$.

- Consider a block \mathcal{B} such that all of its descendants have been already considered. Let \mathcal{B}' be a block in $BC_{\mathcal{B}}(G)$ which has a common cut-vertex x with \mathcal{B} . For any subgraph S of $\mathcal{S}(\mathcal{B})$ which contains x , we keep only the roots r in $root(S)$ such that for any subgraph S' of $\mathcal{S}(\mathcal{B}')$ which contains x , there is at least a root r' in $root(S')$ which gives the compatible values to r .

- We continue this process up to the root R . Suppose that $root(S) \neq \emptyset$ for every subgraph S of the block R . Then by the induction hypothesis, there is at least a root in $root(S)$ for each subgraph S of any block of $BC(G)$ such that the conditions in Lemma 10 are satisfied at any cut-vertex of G . Otherwise, if there exists a subgraph S of the block R such that $root(S) = \emptyset$, then G is not a 4-leaf basic g-network power.

Finally, we can easily choose for each subgraph S of $\mathcal{S}(G)$ a root in $root(S)$ as follows:

- Choose for each subgraph S of R a certain root in $root(S)$.
- Next, for each subgraph S' of $\mathcal{S}(\mathcal{B})$ where \mathcal{B} is a child block of R , choose a certain root in $root(S)$ which is compatible with the chosen roots on R if they share a common cut-vertex. There exists always at least such a root due to the above proof.
- We continue to do that down to the leaf block of $BC(G)$.

Hence, the obtained collection of roots verify the property stated in the theorem. \square

We call each collection of roots calculated in Theorem 4 a **subroot set** of G . So, if \mathcal{R} is a subroot set of G , then each subgraph of $\mathcal{S}(G)$ has exactly one root in \mathcal{R} . Moreover, the cycles in the networks of \mathcal{R} are pairwise vertex disjoint, and the conditions in Lemma 10 are satisfied at any cut-vertex x of G . Therefore, according to Corollary 4 and Theorem 3, if G is a 4-leaf basic g-network power then there is always a basic g-network N containing the subnetworks of \mathcal{R} as induced subgraphs.

Let $\mathcal{R}_c = \{S(c) \mid c \text{ is a cycle in a network of } \mathcal{R}\}$, namely a **cycle-root set** of G .

After constructing the collection of cycles, we consider the remaining parts of the roots in the next section.

4.5 Putting the things together and constructing the roots

Let N be a basic g-network without invisible vertices. Denote by $X(N)$ the graph obtained from N by labelling each inner vertex following the label of its leaf and removing all of the leaves. So X is a bijection from the set of basic g-networks without invisible vertices to the set of g-networks. It is easy to see that the 4-leaf power of N is exactly the square of $X(N)$.

Proposition 3 *Let H be the square of a tree and its vertices are distinctly labelled.*

If H is a clique then the square tree roots of H are all possible stars having vertices set equal to the vertices set of H .

If H contains 2 maximal cliques then H has two square tree roots, each one consisting of two adjacent stars. These roots have the same forms but are labelled differently.

If H contains at least 3 maximal cliques then H has a unique square tree root.

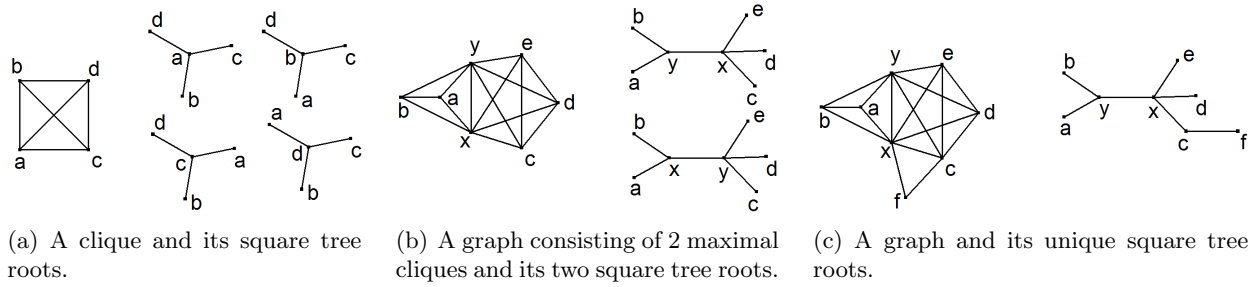


Figure 24: Tree square roots.

Proof: If H is a clique, it is easy to see that its tree roots must be stars. These stars are different by the labels of their middle vertices (Figure 24(a)).

[4] represents a characteristic of the square of a tree and the way to construct its roots: each maximal clique corresponds to a star, and the middle vertex of a star is uniquely determined (Figure 24(c)) except when H contains only two maximal cliques. For example H has 2 maximal cliques A, B , so they have exactly 2 common vertices x, y . So, the star corresponding to A can take either x or y as its middle vertex, and the star corresponding to B take the other vertex as its middle vertex. Hence, in this case H has two roots (Figure 24(b)). \square

Given a cycle-root set \mathcal{R}_c of G , and a block \mathcal{B} of G , denote by $\mathcal{R}_c[\mathcal{B}]$ the cycles of \mathcal{R}_c restricted on \mathcal{B} .

Definition 5 *Let \mathcal{B} be a block of G and let \mathcal{R}_c be a cycle-root set of G . We define $\mathcal{B} \setminus \mathcal{R}_c$ as the graph such that $MC(\mathcal{B} \setminus \mathcal{R}_c)$ is constructed as follows:*

For each cycle c in $\mathcal{R}_c[\mathcal{B}]$, let A_1, \dots, A_k be the nodes of $MC(\mathcal{B})$ corresponding to the visible, invisible (quasi) stars of $S(c)$. Let M be the node corresponding to the subnetwork of \mathcal{F} contains c if there is any. Then, delete M and all edges between A_i from $MC(\mathcal{B})$.

Example 5 *For instance, see the two examples in Figures 25, 26. In Figure 25, $MC(\mathcal{B})$ contains a cycle which is a subgraph of $\mathcal{C}'(\mathcal{B})$. By Lemma 5, there is a cycle in any root of \mathcal{B} in which each node A, B, C, D, E, F corresponds to a visible or invisible star of this cycle. So, in $MC(\mathcal{B} \setminus \mathcal{R}_c)$, we delete the edges between them. In Figure 26, $MC(\mathcal{B})$ contains subgraph AMB in $\mathcal{C}_2(\mathcal{B})$. By Lemma 6, there is a cycle in any root of \mathcal{B} in which each node A, B corresponds to a visible star and M corresponds to the cycle. So, in $MC(\mathcal{B} \setminus \mathcal{R}_c)$, we delete M and the edges MA, MB .*

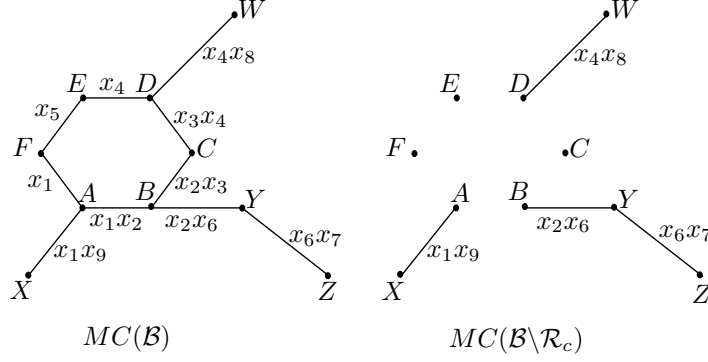


Figure 25: Example 1

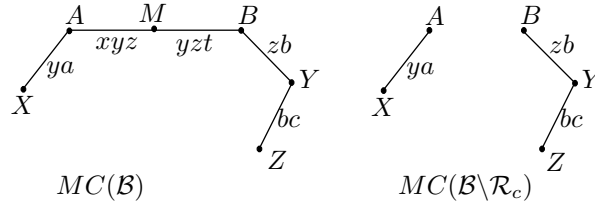
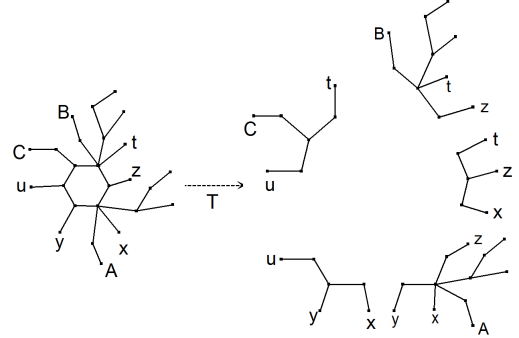


Figure 26: Example 2

Definition 6 (Separating the stars) Let c be a cycle in a network N . The operation of **separating the stars** of $S(c)$ is illustrated in the figure, i.e. the stars which have the middle vertices on c are separated. Denote by $T(N)$ the graph obtained from N by this operation on all cycles of N .

So, for any network N , $T(N)$ is a forest. By Lemma 3-(i), if N has an invisible vertex, then it must be contained in a cycle of $\mathcal{R}_c[\mathcal{B}_i]$.

So, each tree of $T(N)$ is either an invisible star or a tree without invisible vertex.



Theorem 5 Let $\mathcal{B}_1, \dots, \mathcal{B}_k$ be the blocks of G , then G is a 4-leaf basic g -network power iff G has a cycle-root set \mathcal{R}_c such that for every $1 \leq i \leq k$, $\mathcal{B}_i \setminus \mathcal{R}_c$ is the square of a forest.

Proof: \Rightarrow Suppose that G has a 4-leaf basic g -network root N . Then, the collection of cycles of N forms a cycle-root set \mathcal{R}_c of G . For each block \mathcal{B}_i of G , let $N_i = N[\mathcal{B}_i]$.

If \mathcal{B}_i is a clique then $N[\mathcal{B}_i]$ is an invisible star, i.e. it does not contain any cycle. So, $\mathcal{B} \setminus \mathcal{R}_c = \mathcal{B}$, which is obviously the square of a tree.

Otherwise, for any $i = 1, \dots, k$, we calculate $T(N_i)$. By Definitions 5 and 6, each connected component H of $\mathcal{B}_i \setminus \mathcal{R}_c$ is a 4-leaf power of a tree T in the forest $T(N_i)$. If T is an invisible star, then H is a clique, so it is obviously the square of a tree. Otherwise, T is without invisible vertex, then H has $X(T)$ as a square tree root. In other words, $\mathcal{B}_i \setminus \mathcal{R}_c$ is the square of a forest.

\Leftarrow Suppose that G has a cycle-root set \mathcal{R}_c such that $\mathcal{B}_i \setminus \mathcal{R}_c$ is the square of a forest for any i .

If \mathcal{B}_i is a clique, then we construct N_i as an invisible star having the leaf set equal to the vertex set of \mathcal{B}_i .

Otherwise, we will show a way to construct a network N_i which is a 4-leaf basic g-network root of \mathcal{B}_i . Let H be a connected component of $\mathcal{B}_i \setminus \mathcal{R}_c$, so H is the square of a tree. There are two cases:

- H is a clique which corresponds to an invisible (quasi) star having the middle vertex on a cycle of $\mathcal{R}_c[\mathcal{B}_i]$. In this case we construct a 4-leaf root r of H which is an invisible star. It is the case of the clique (u, t) in Figure 4.

- otherwise, we will construct a 4-leaf root r of H which is a basic tree without invisible vertex. It is equivalent with constructing a square tree root $t = X^{-1}(r)$ of H . The root r of H must be *compatible* with the cycles in $\mathcal{R}_c[\mathcal{B}_i]$, it means that it must satisfy the following conditions: For any maximal clique A of H such that there is a cycle c of $\mathcal{R}_c[\mathcal{B}_i]$ and $S(c)[A]$ is a visible (quasi) star, then $r[A]$ must be the visible star having the same label of the middle vertex of $S(c)[A]$. In other words, $t[A]$ is a star having the same label of the middle vertex of $S(c)[A]$.

For example, in Figure 4, let consider the connected component H which contains the maximal clique $A = (b, c, t, z)$ of $\mathcal{B}_i \setminus \mathcal{R}_c$. There is a cycle c in $\mathcal{R}_c[\mathcal{B}_i]$ where $S(c)[A]$ is a visible star having t as the label of the middle vertex. So, we construct the root of H as a visible star also having t as the label of the middle vertex.

There is always such a root of H because:

- . If H is a clique, i.e. $H = A$, then $S(c)[A]$ is certainly a root of H . So we need only to choose $r = S(c)[A]$.

- . If H contains 2 maximal cliques A, B , then the label of the middle vertex of $S(c)[A]$ is one of the common labels of A, B (see Remark 1). It is also the way that we choose the middle vertex for the root of H in this case (see Proposition 3).

- . If H contains at least 3 maximal cliques, then by Proposition 3, it has a unique square root t . The label of the middle vertex of $t[A]$, or $r[A]$, and $S(c)[A]$ must have the same label. This claim is followed from the fact that when we construct the roots for each subgraph S in $\mathcal{S}(\mathcal{B})$, we consider also the nodes around S (see the way of determining the middle vertex in Remark 1 and of considering the nodes around each subgraph up to distance 2 in Definition 4).

Denote by F_i the forest consisting of the roots of $\mathcal{B}_i \setminus \mathcal{R}_c$ such that the root of each connected component of $\mathcal{B}_i \setminus \mathcal{R}_c$ is constructed as above. We calculate N_i by overlaying the common parts of the stars in the trees of F_i . In fact, $N_i = T^{-1}(F_i)$, because this is the inverse operation of separating the cycles in Definition 6. It is easy to see that N_i is a 4-leaf root of \mathcal{B}_i , and each cycle in $\mathcal{R}_c[\mathcal{B}_i]$ is a cycle of N_i .

We must prove furthermore that N_i is a basic g-network. For each block \mathcal{B}_i , each cycle of $\mathcal{R}_c[\mathcal{B}_i]$ is a cycle of N_i , so these cycles are vertex disjoint. Suppose that N_i has a cycle c not in $\mathcal{R}_c[\mathcal{B}_i]$. N_i is a 4-leaf root of \mathcal{B}_i , so by Corollary 5, c must be contained in a root of a subgraph S of $\mathcal{S}(\mathcal{B}_i)$. However, by definition of cycle-root set, S must has a root r whose cycle is contained in $\mathcal{R}_c[\mathcal{B}_i]$. This cycle must be exactly c because otherwise r has two distinct cycles, a contradiction. Hence, N_i is a basic g-network.

Finally, we combine N_i for any $i = 1, \dots, k$ by gluing all $p(x)$ for any cut-vertex x of G and deleting all copies of x except one. So, the resulting network N is a 4-leaf basic g-network root of G due to Lemma 10 and the property of the cycle-root set \mathcal{R}_c , i.e. no cycles in the roots of two different blocks of G can intersect. \square

We can resume the method to construct a 4-leaf basic g-network root of G in the following algorithm. See Figure 4 for an illustration.

<p>Data: A connected undirected graph G, and a fixed k</p> <p>Result: A 4-leaf basic g-network root of G, if there exists one</p> <pre> 1 for (each block \mathcal{B} of G) do 2 Calculate $MC(\mathcal{B})$; 3 Calculate $\mathcal{S}(\mathcal{B}) = \mathcal{C}_e(\mathcal{B}) \cup \mathcal{C}_o(\mathcal{B}) \cup \mathcal{C}_c(\mathcal{B})$; 4 Construct the root set $r(S)$ of each subgraph $S \in \mathcal{S}(\mathcal{B})$; 5 if ($\exists S \in \mathcal{S}(\mathcal{B})$ such that $r(S) = \emptyset$) then 6 return null; 7 if (G does not have any cycle-root set) then 8 return null; 9 Calculate a cycle-root set \mathcal{R}_c of G. 10 for (each block \mathcal{B} of G) do 11 Calculate $MC(\mathcal{B} \setminus \mathcal{R}_c)$ and $\mathcal{B} \setminus \mathcal{R}_c$; 12 if ($\mathcal{B} \setminus \mathcal{R}_c$ is not the square of a forest) then 13 return null; 14 Construct the square root of each connected component of $\mathcal{B} \setminus \mathcal{R}_c$ compatible with \mathcal{R}_c; 15 Construct a 4-leaf basic g-network root of \mathcal{B}; 16 $N \leftarrow$ Combine the 4-leaf basic g-network root of each block; 17 return N </pre>
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Algorithm 1: Construction of 4-leaf basic g-network root

Corollary 6 *Recognizing a 4-leaf power of a basic g-network and constructing one of its roots, if there is any, can be done in polynomial time.*

Proof: We will prove that Algorithm 1 is polynomial.

Calculating the blocks of G is done in linear time.

For each block \mathcal{B} of G , let $V_{\mathcal{B}}, E_{\mathcal{B}}$ be the number of vertices and edges of \mathcal{B} . We know that $MC(\mathcal{B})$ is calculated in $|V_{\mathcal{B}}| \cdot |E_{\mathcal{B}}|^2$ (Theorem 2) and $BC(\mathcal{B})$ is constructible in linear time $O(|V_{\mathcal{B}}| + |E_{\mathcal{B}}|)$.

We consider now the complexity to calculate the subgraphs of $\mathcal{S}(\mathcal{B})$. It can be deduced from Lemmas 5, 6 and the lists of roots of C_6, C_8 that the subgraphs of $\mathcal{C}'(\mathcal{B}), \mathcal{C}_1(\mathcal{B}), \mathcal{C}_3(\mathcal{B}), \mathcal{C}_6(\mathcal{B}), \mathcal{C}_8(\mathcal{B})$ must be pairwise node disjoint, otherwise the cycles in their roots intersect. Moreover, they are the only subgraphs of $\mathcal{S}(\mathcal{B})$ that contain cycles. It implies that if \mathcal{B} is a 4-leaf basic g-network power, then each non trivial block of $MC(\mathcal{B})$ is a one of these subgraphs.

Remark that:

- Each subgraph of $\mathcal{C}'(\mathcal{B}), \mathcal{C}_1(\mathcal{B}), \mathcal{C}_6(\mathcal{B}), \mathcal{C}_8(\mathcal{B})$ is a cycle,
- Each subgraph of $\mathcal{C}_3(\mathcal{B})$ contains 8 nodes, 12 edges,
- The number of nodes of $MC(\mathcal{B})$ is bounded by $3 \cdot |E_{\mathcal{B}}|$ (Corollary 3).

So if \mathcal{B} is a 4-leaf basic g-network power, then the number of edges of $MC(\mathcal{B})$ is bounded by $c \cdot |E_{\mathcal{B}}|$, where c is a constant. Hence, calculating the blocks of $MC(\mathcal{B})$ and checking the form of each one can be done in $O(|E_{\mathcal{B}}|)$.

Calculating $\mathcal{C}_2(\mathcal{B})$ is also done in $O(|E_{\mathcal{B}}|)$.

Calculating the other subgraphs of $\mathcal{C}_o(\mathcal{B})$, $\mathcal{C}_c(\mathcal{B})$ is done in $O(|E_{\mathcal{B}}|^4)$ because each subgraph has at most 4 nodes. So, it takes in total $O(|E_{\mathcal{B}}|^4)$ to construct $\mathcal{S}(\mathcal{B})$.

Next, constructing the root set of each subgraph is done in at most $O(|E_{\mathcal{B}}|)$ following the constructible Lemmas 5, 6 for the subgraphs in $\mathcal{C}_c(\mathcal{B})$, and following the provided finite list of roots, the construction of $r(S)$ in Definition 4 for the subgraphs in $\mathcal{C}_o(\mathcal{B}) \cup \mathcal{C}_c(\mathcal{B})$. There are at most $O(|E_{\mathcal{B}}|)$ subgraphs, then in total it takes $O(|E_{\mathcal{B}}|^2)$ to find the roots set for each subgraph of $\mathcal{S}(\mathcal{B})$.

By Theorem 4, applying the condition between two non-disjoint blocks in order to calculate a cycle-root set \mathcal{R}_c is done in linear time $O(|V_{\mathcal{B}}| + |E_{\mathcal{B}}|)$.

Calculating $\mathcal{B} \setminus \mathcal{R}_c$, determining if it is the square of a forest [11, 5] and constructing a root of it by Theorem 5 are done in linear time $O(|V_{\mathcal{B}}| + |E_{\mathcal{B}}|)$.

Finally, combining the root of the blocks of G is done in linear time $O(|V| + |E|)$.

Then, we have the total complexity is $O(|V| \cdot |E|^2 + |E|^4) = O(|E|^4)$ (because G is connected).

□

Corollary 7 *Recognizing a square of a basic g-network and constructing one of its root, if there is any, can be done in polynomial time.*

Proof: It is easily to see that G is the square of a basic g-network iff it is the 4-leaf power of a basic g-network without invisible vertex. So, it can be deduced from Theorem 5 that G is the square of a basic g-network iff G is biconnected, has a cycle-root set \mathcal{R}_c such that the cycles in \mathcal{R}_c are without invisible vertex, and $G \setminus \mathcal{R}_c$ is the square of a forest. So, any cycle of $\mathcal{C}'(G)$ must have all edges of weight 2. We do not need to calculate $\mathcal{C}_i(G)$ for $i = 4, \dots, 8$ because all roots of these subgraphs contain at least one invisible vertex. We do not neither have to calculate $\mathcal{C}_c(G)$ because G is biconnected. So, the complexity of this problem is equal to the complexity of calculating the maximal cliques graph, which is $O(|V| \cdot |E|^2)$. □

5 Conclusions and Perspectives

Of course the complexity of our algorithm could be improved, but we hope that these partial structural results on k -leaf \mathcal{N} network power could help to answer the biological question described in the introduction. We have proven here that the recognition of 4-leaf g-network power graphs is polynomial. It would be natural to enlarge this result to the class of extended Cactus graphs (with leaves but for which two cycles share at most one vertex).

But we propose a stronger conjecture in the same flavour that [9]:

Conjecture: The recognition of 4-leaf \mathcal{N} power graphs is polynomial for every bounded treewidth \mathcal{N} class of phylogenetic networks.

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